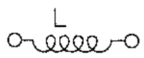
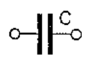
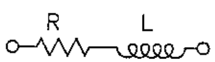
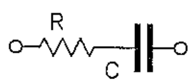
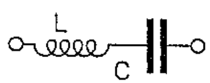
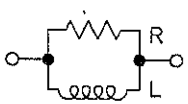
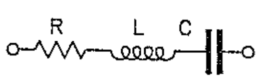
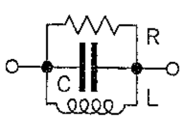
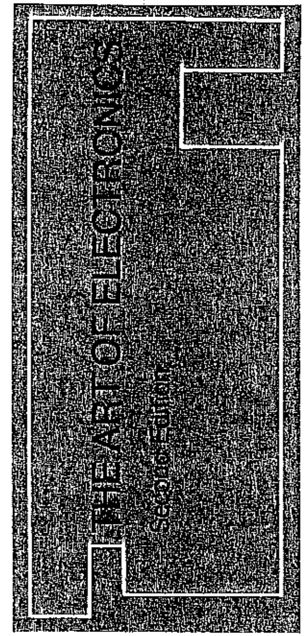
Circuit	Impedance Operator	Admittance Operator
	R	$\frac{1}{R}$
	$j\omega L$	$\frac{1}{j\omega L}$
	$\frac{1}{j\omega C}$	$j\omega C$
	$R + j\omega L$	$\frac{R - j\omega L}{R^2 + \omega^2 L^2}$
	$R + \frac{1}{j\omega C}$	$\frac{R + \frac{j}{\omega C}}{R^2 + \frac{1}{\omega^2 C^2}}$
	$j\left(\omega L - \frac{1}{\omega C}\right)$	$\frac{j}{\frac{1}{\omega C} - \omega L}$
	$\frac{\frac{1}{R} + \frac{1}{j\omega L}}{\frac{1}{R^2} + \frac{1}{\omega^2 L^2}}$	$\frac{1}{R} - \frac{j}{\omega L}$
	$R + j\left(\omega L - \frac{1}{\omega C}\right)$	–
	–	$\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$

Table 3.1: Some admittance and impedance operators.



1.13 *RC* circuits: *V* and *I* versus time

When dealing with ac circuits (or, in general, any circuits that have changing voltages and currents), there are two possible approaches. You can talk about V and I versus time, or you can talk about amplitude versus signal frequency. Both approaches have their merits, and you find yourself switching back and forth according to which description is most convenient in each situation. We will begin our study of ac circuits in the time domain. Beginning with Section 1.18, we will tackle the frequency domain.

What are some of the features of circuits with capacitors? To answer this question, let's begin with the simple RC circuit (Fig. 1.29). Application of the capacitor rules gives

$$C\frac{dV}{dt} = I = -\frac{V}{R}$$

This is a differential equation, and its solution is

$$V = Ae^{-t/RC}$$

So a charged capacitor placed across a resistor will discharge as in Figure 1.30.

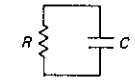


Figure 1.29

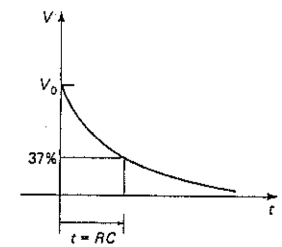


Figure 1.30. RC discharge waveform.

Time constant

The product RC is called the *time constant* of the circuit. For R in ohms and C in farads, the product RC is in seconds. A microfarad across 1.0k has a time constant of 1ms; if the capacitor is initially charged to 1.0 volt, the initial current is 1.0mA.

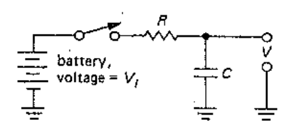


Figure 1.31

Figure 1.31 shows a slightly different circuit. At time $t = 0$, someone connects the battery. The equation for the circuit is then

$$I = C\frac{dV}{dt} = \frac{V_i - V}{R}$$

with the solution

$$V = V_i + Ae^{-t/RC}$$

(Please don't worry if you can't follow the mathematics. What we are doing is getting some important results, which you should remember. Later we will use the results often, with no further need for the mathematics used to derive them.) The constant A is determined by initial conditions (Fig. 1.32): $V = 0$ at $t = 0$; therefore, $A = -V_i$, and

$$V = V_i(1 - e^{-t/RC})$$

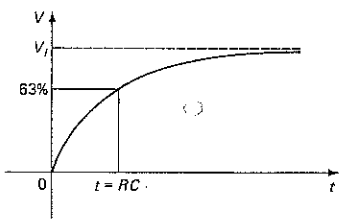


Figure 1.32

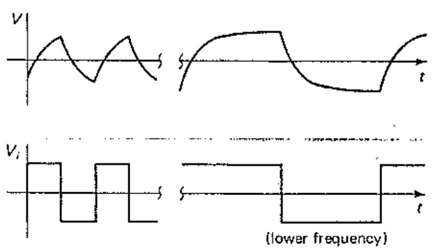


Figure 1.33. Output (top waveform) across a capacitor, when driven by square waves through a resistor.

Decay to equilibrium

Eventually (when $t \gg RC$), V reaches V_i. (Presenting the "$5RC$ rule of thumb": a capacitor charges or decays to within 1% of its final value in 5 time constants.) If we then change V_0 to some other value (say, 0), V will decay toward that new value with an exponential $e^{-t/RC}$. For example, a square-wave input for V_0 will produce the output shown in Figure 1.33.

EXERCISE 1.13
Show that the rise time (the time required to go from 10% to 90% of its final value) of this signal is $2.2RC$.

You might ask the obvious next question: What about $V(t)$ for arbitrary $V_i(t)$? The solution involves an inhomogeneous differential equation and can be solved by standard methods (which are, however, beyond the scope of this book). You would find

$$V(t) = \frac{1}{RC}\int_{-\infty}^{t} V_i(\tau)e^{-(t-\tau)/RC}d\tau$$

That is, the RC circuit averages past history at the input with a weighting factor $e^{-\Delta t/RC}$.

In practice, you seldom ask this question. Instead, you deal in the *frequency domain* and ask how much of each frequency component present in the input gets through. We will get to this important topic soon (Section 1.18). Before we do, though, there are a few other interesting circuits we can analyze simply with this time-domain approach.

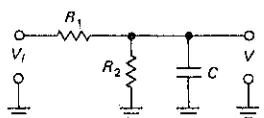


Figure 1.34

Simplification by Thévenin equivalents

We could go ahead and analyze more complicated circuits by similar methods, writing down the differential equations and trying to find solutions. For most purposes it simply isn't worth it. This is as complicated an RC circuit as we will need. Many other circuits can be reduced to it (e.g., Fig. 1.34). By just using the Thévenin equivalent of the voltage divider formed by R_1 and R_2, you can find the output $V(t)$ produced by a step input for V_0.

EXERCISE 1.14
$R_1 = R_2 = 10$k, and $C = 0.1\mu$F in the circuit shown in Figure 1.34. Find $V(t)$ and sketch it.

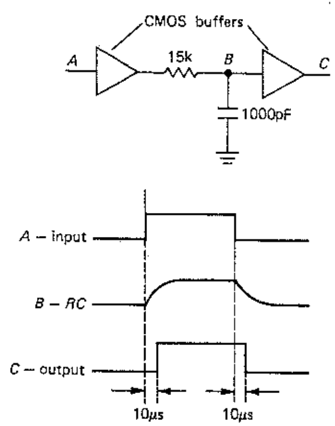


Figure 1.35. Producing a delayed digital waveform with the help of an RC.

Example: time-delay circuit

We have already mentioned logic levels, the voltages that digital circuits live on. Figure 1.35 shows an application of capacitors to produce a delayed pulse. The triangular symbols are "CMOS buffers." They give a HIGH output if the input is HIGH (more than one-half the dc power-supply voltage used to power them), and vice versa. The first buffer provides a replica of the input signal, but with low source resistance, and prevents input loading by the RC (recall our earlier discussion of circuit loading in Section 1.05). The RC output has the characteristic decays and causes the output buffer to switch 10μs after the input transitions (an RC reaches 50% output in $0.7RC$). In an actual application you would have to consider the effect of the buffer input threshold deviating from one-half the supply voltage, which would alter the delay and change the output pulse width. Such a circuit is sometimes used to delay a pulse so that something else can happen first. In designing circuits you try not to rely on tricks like this, but they're occasionally handy.

1.14 Differentiators

Look at the circuit in Figure 1.36. The voltage across C is $V_{in} - V$, so

$$I = C\frac{d}{dt}(V_{in} - V) = \frac{V}{R}$$

If we choose R and C small enough so that $dV/dt \ll dV_{in}/dt$, then

$$C\frac{dV_{in}}{dt} \approx \frac{V}{R}$$

or

$$V(t) = RC\frac{d}{dt}V_{in}(t)$$

That is, we get an output proportional to the rate of change of the input waveform.

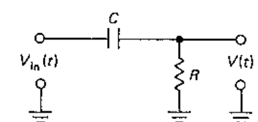


Figure 1.36

To keep $dV/dt \ll dV_{in}/dt$, we make the product RC small, taking care not to "load" the input by making R too small (at the transition the change in voltage across the capacitor is zero, so R is the load seen by the input). We will have a better criterion for this when we look at things in the frequency domain. If you drive this circuit with a square wave, the output will be as shown in Figure 1.37.

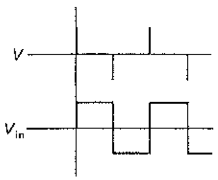


Figure 1.37. Output waveform (top) from differentiator driven by a square wave.

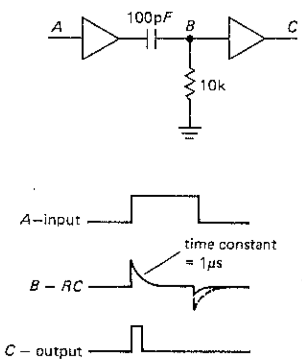


Figure 1.38. Leading-edge detector.

Differentiators are handy for detecting *leading edges* and *trailing edges* in pulse signals, and in digital circuitry you sometimes see things like those depicted in Figure 1.38. The RC differentiator generates spikes at the transitions of the input signal, and the output buffer converts the spikes to short square-topped pulses. In practice, the negative spike will be small because of a diode (a handy device discussed in Section 1.25) built into the buffer.

Unintentional capacitive coupling

Differentiators sometimes crop up unexpectedly, in situations where they're not welcome. You may see signals like those shown in Figure 1.39. The first case is caused by a square wave somewhere in the circuit coupling capacitively to the signal line you're looking at; that might indicate a missing resistor termination on your signal line. If not, you must either reduce the source resistance of the signal line or find a way to reduce capacitive coupling from the offending square wave. The second case is typical of what you might see when you look at a square wave, but have a broken connection somewhere, usually at the scope probe. The very small capacitance of the broken connection combines with the scope input resistance to form a differentiator. *Knowing that you've got a differentiated "something" can help you find the trouble and eliminate it.*

Figure 1.39

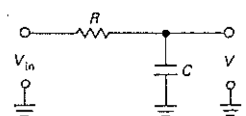


Figure 1.40

1.15 Integrators

Take a look at the circuit in Figure 1.40. The voltage across R is $V_{in} - V$, so

$$I = C\frac{dV}{dt} = \frac{V_{in} - V}{R}$$

If we manage to keep $V \ll V_{in}$, by keeping the product RC large, then

$$C\frac{dV}{dt} \approx \frac{V_{in}}{R}$$

or

$$V(t) = \frac{1}{RC}\int^{t} V_{in}(t)\,dt + \text{constant}$$

We have a circuit that performs the integral over time of an input signal! You can see how the approximation works for a square-wave input: $V(t)$ is then the exponential charging curve we saw earlier (Fig. 1.41). The first part of the exponential is a ramp, the integral of a constant; as we increase the time constant RC, we pick off a smaller part of the exponential, i.e., a better approximation to a perfect ramp.

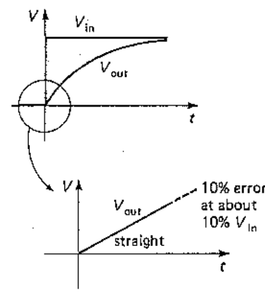


Figure 1.41

Note that the condition $V \ll V_{in}$ is just the same as saying that I is proportional to V_{in}. If we had as input a *current* $I(t)$, rather than a voltage, we would have an exact integrator. A large voltage across a large resistance approximates a current source and, in fact, is frequently used as one.

Later, when we get to operational amplifiers and feedback, we will be able to build integrators without the restriction $V_{out} \ll V_{in}$. They will work over large frequency and voltage ranges with negligible error.

The integrator is used extensively in analog computation. It is a useful subcircuit that finds application in control systems, feedback, analog/digital conversion, and waveform generation.

Ramp generators

At this point it is easy to understand how a ramp generator works. This nice circuit is extremely useful, for example in timing circuits, waveform and function generators, oscilloscope sweep circuits, and analog/digital conversion circuitry. The circuit uses a constant current to charge a capacitor (Fig. 1.42). From the capacitor equation $I = C(dV/dt)$, you get $V(t) = (I/C)t$. The output waveform is as shown in Figure 1.43. The ramp stops when the current source "runs out of voltage," i.e., reaches the limit of its compliance. The curve for a simple RC, with the resistor tied to a voltage source equal to the compliance of the current source, and with R chosen so that the current at zero output voltage is the same as that of the current source, is also drawn for comparison. (Real current sources generally have output compliances limited by the power-supply voltages used in making them, so the comparison is realistic.) In the next chapter, which deals with transistors, we will design some current sources, with some refinements to follow in the chapters on operational amplifiers (op-amps) and field-effect transistors (FETs). Exciting things to look forward to!

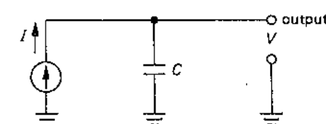


Figure 1.42. A constant current source charging a capacitor generates a ramp voltage waveform.

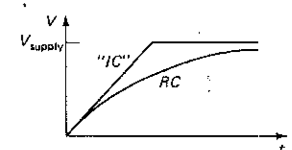


Figure 1.43

EXERCISE 1.15
A current of 1mA charges a 1µF capacitor. How long does it take the ramp to reach 10 volts?

INDUCTORS AND TRANSFORMERS

1.16 Inductors

If you understand capacitors, you won't have any trouble with inductors (Fig. 1.44). They're closely related to capacitors; the rate of current change in an inductor depends on the voltage applied across it, whereas the rate of voltage change in a capacitor depends on the current through it. The defining equation for an inductor is

$$V = L\frac{dI}{dt}$$

where L is called the *inductance* and is measured in henrys (or mH, μH, etc.). Putting a voltage across an inductor causes the current to rise as a ramp (for a capacitor, supplying a constant current causes the voltage to rise as a ramp); 1 volt across 1 henry produces a current that increases at 1 amp per second.

Figure 1.44. Inductor.

As with capacitive current, inductive current is not simply proportional to voltage. Furthermore, unlike the situation in a resistor, the power associated with inductive current (V times I) is not turned into heat, but is stored as energy in the inductor's magnetic field. You get all that energy back when you interrupt the inductor's current.

The symbol for an inductor looks like a coil of wire; that's because, in its simplest form, that's all it is. Variations include coils wound on various core materials, the most popular being iron (or iron alloys, laminations, or powder) and ferrite, a black, nonconductive, brittle magnetic material. These are all ploys to multiply the inductance of a given coil by the "permeability" of the core material. The core may be in the shape of a rod, a toroid (doughnut), or even more bizarre shapes, such as a "pot core" (which has to be seen to be understood; the best description we can think of is a doughnut mold split horizontally in half, if doughnuts were made in molds).

Inductors find heavy use in radiofrequency (RF) circuits, serving as RF "chokes" and as parts of tuned circuits (see Chapter 13). A pair of closely coupled inductors forms the interesting object known as a transformer. We will talk briefly about them in the next section.

An inductor is, in a real sense, the opposite of a capacitor. You will see how that works out in the next few sections of this chapter, which deal with the important subject of *impedance.*

1.17 Transformers

A transformer is a device consisting of two closely coupled coils (called primary and secondary). An ac voltage applied to the primary appears across the secondary, with a voltage multiplication proportional to the turns ratio of the transformer and a current multiplication inversely proportional to the turns ratio. Power is conserved. Figure 1.45 shows the circuit symbol for a laminated-core transformer (the kind used for 60Hz ac power conversion).

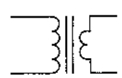


Figure 1.45. Transformer.

Transformers are quite efficient (output power is very nearly equal to input power); thus, a step-up transformer gives higher voltage at lower current. Jumping ahead for a moment, a transformer of turns ratio n increases the impedance by n^2. There is very little primary current if the secondary is unloaded.

Transformers serve two important functions in electronic instruments: They change the ac line voltage to a useful (usually lower) value that can be used by the circuit, and they "isolate" the electronic device from actual connection to the power line, because the windings of a transformer are electrically insulated from each other. *Power transformers* (meant for use from the 110V power line) come in an enormous variety of secondary voltages and currents: outputs as low as 1 volt or so up to several thousand volts, current ratings from a few milliamps to hundreds of amps. Typical transformers for use in electronic instruments might have secondary voltages from 10 to 50 volts, with current ratings of 0.1 to 5 amps or so.

Transformers for use at audiofrequencies and radiofrequencies are also available. At radiofrequencies you sometimes use tuned transformers, if only a narrow range of frequencies is present. There is also an interesting class of transmission-line transformer that we will discuss briefly in Section 13.10. In general, transformers for use at high frequencies must use special core materials or construction to minimize core losses, whereas low-frequency transformers (e.g., power transformers) are burdened instead by large and heavy cores. The two kinds of transformers are in general not interchangeable.

IMPEDANCE AND REACTANCE

Warning: This section is somewhat mathematical; you may wish to skip over the mathematics, but be sure to pay attention to the results and graphs.

Circuits with capacitors and inductors are more complicated than the resistive circuits we talked about earlier, in that their behavior depends on frequency: A "voltage divider" containing a capacitor or inductor will have a frequency-dependent division ratio. In addition, circuits containing these components (known collectively as *reactive* components) "corrupt" input waveforms such as square waves, as we just saw.

However, both capacitors and inductors are *linear* devices, meaning that the amplitude of the output waveform, whatever its shape, increases exactly in proportion to the input waveform's amplitude. This linearity has many consequences, the most important of which is probably the following: *The output of a linear circuit, driven with a sine wave at some frequency* f, *is itself a sine wave at the same frequency (with, at most, changed amplitude and phase).*

Because of this remarkable property of circuits containing resistors, capacitors, and inductors (and, later, linear amplifiers), it is particularly convenient to analyze any such circuit by asking how the output voltage (amplitude and phase) depends on the input voltage, *for sine-wave input at a single frequency*, even though this may not be the intended use. A graph of the resulting *frequency response*, in which the ratio of output to input is plotted for each sine-wave frequency, is useful for thinking about many kinds of waveforms. As an example, a certain "boom-box" loudspeaker might have the frequency response shown in Figure 1.46, where the "output" in this case is of course sound pressure, not voltage. It is desirable for a speaker to have a "flat" response, meaning that the graph of sound pressure versus frequency is constant over the band of audible frequencies. In this case the speaker's deficiencies can be corrected by introducing a passive filter with the inverse response (as shown) into the amplifiers of the radio.

As we will see, it is possible to generalize Ohm's law, replacing the word "resistance" with "impedance," in order to describe any circuit containing these linear passive devices (resistors, capacitors, and inductors). You could think of the subject of impedance and reactance as Ohm's law for circuits that include capacitors and inductors. Some important terminology: Impedance is the "generalized resistance"; inductors

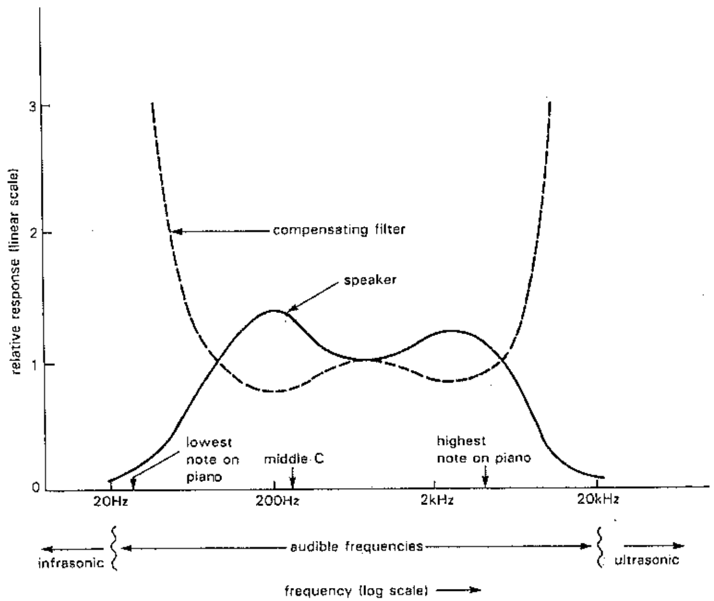


Figure 1.46. Example of frequency analysis: "boom box" loudspeaker equalization.

and capacitors have *reactance* (they are "reactive"); resistors have *resistance* (they are "resistive"). In other words, impedance = resistance + reactance (more about this later). However, you'll see statements like "the impedance of the capacitor at this frequency is ... " The reason you don't have to use the word "reactance" in such a case is that impedance covers everything. In fact, you frequently use the word "impedance" even when you know it's a resistance you're talking about; you say "the source impedance" or "the output impedance" when you mean the Thévenin equivalent resistance of some source. The same holds for "input impedance."

In all that follows, we will be talking about circuits driven by sine waves at a single frequency. Analysis of circuits driven by complicated waveforms is more elaborate, involving the methods we used earlier (differential equations) or decomposition of the waveform into sine waves (Fourier analysis). Fortunately, these methods are seldom necessary.

1.18 Frequency analysis of reactive circuits

Let's start by looking at a capacitor driven by a sine-wave voltage source (Fig. 1.47). The current is

$$I(t) = C\frac{dV}{dt} = C\omega V_0 \cos \omega t$$

i.e., a current of amplitude I, with the phase leading the input voltage by 90°. If we consider amplitudes only, and disregard phases, the current is

$$I = \frac{V}{1/\omega C}$$

(Recall that $\omega = 2\pi f$.) It behaves like a frequency-dependent resistance $R = 1/\omega C$, but in addition the current is 90° out of phase with the voltage (Fig. 1.48).

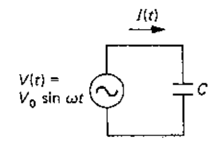


Figure 1.47

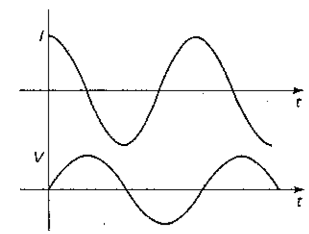


Figure 1.48

For example, a 1μF capacitor put across the 110 volt (rms) 60Hz power line draws a current of rms amplitude

$$I = \frac{110}{1/(2\pi \times 60 \times 10^{-6})} = 41.5\text{mA (rms)}$$

Note: At this point it is necessary to get into some complex algebra; you may wish to skip over the math in some of the following sections, taking note of the results as we derive them. A knowledge of the detailed mathematics is not necessary in order to understand the remainder of the book. Very little mathematics will be used in later chapters. The section ahead is easily the most difficult for the reader with little mathematical preparation. Don't be discouraged!

Voltages and currents as complex numbers

As you have just seen, there can be phase shifts between the voltage and current in an ac circuit being driven by a sine wave at some frequency. Nevertheless, as long as the circuit contains only *linear* elements (resistors, capacitors, inductors), the magnitudes of the currents everywhere in the circuit are still proportional to the magnitude of the driving voltage, so we might hope to find some generalization of voltage, current, and resistance in order to rescue Ohm's law. Obviously a single number won't suffice to specify the current, say, at some point in the circuit, because we must somehow have information about both the magnitude and phase shift.

Although we can imagine specifying the magnitudes and phase shifts of voltages and currents at any point in the circuit by writing them out explicitly, e.g., $V(t) = 23.7\sin(377t + 0.38)$, it turns out that our requirements can be met more simply by using the algebra of complex numbers to *represent* voltages and currents. Then we can simply add or subtract the complex number representations, rather than laboriously having to add or subtract the actual sinusoidal functions of time themselves. Because the actual voltages and currents are real quantities that vary with time, we must develop a rule for converting from actual quantities to their representations, and vice versa. Recalling once again that we are talking about a single sine-wave frequency, ω, we agree to use the following rules:

1. Voltages and currents are *represented* by the complex quantities **V** and **I**. The voltage $V_0\cos(\omega t+\phi)$ is to be represented by the complex number $V_0e^{j\phi}$. Recall that $e^{j\theta} = \cos\theta + j\sin\theta$, where $j = \sqrt{-1}$.

2. Actual voltages and currents are obtained by multiplying their complex number representations by $e^{j\omega t}$ and then taking the real part: $V(t) = \mathcal{R}e(\mathbf{V}e^{j\omega t})$, $I(t) = \mathcal{R}e(\mathbf{I}e^{j\omega t})$

In other words,

circuit voltage versus time		complex number representation
$V_0\cos(\omega t + \phi)$	⟶ / ⟵ multiply by $e^{j\omega t}$ and take real part	$V_0e^{j\phi} = a + jb$

(In electronics, the symbol j is used instead of i in the exponential in order to avoid confusion with the symbol i meaning current.) Thus, in the general case the actual voltages and currents are given by

$$V(t) = \mathcal{R}e(\mathbf{V}e^{j\omega t}) = \mathcal{R}e(\mathbf{V})\cos\omega t - \mathcal{I}m(\mathbf{V})\sin\omega t$$

$$I(t) = \mathcal{R}e(\mathbf{I}e^{j\omega t}) = \mathcal{R}e(\mathbf{I})\cos\omega t - \mathcal{I}m(\mathbf{I})\sin\omega t$$

For example, a voltage whose complex representation is

$$\mathbf{V} = 5j$$

corresponds to a (real) voltage versus time of

$$V(t) = \mathcal{R}e[5j\cos\omega t + 5j(j)\sin\omega t] = -5\sin\omega t \text{ volts}$$

Reactance of capacitors and inductors

With this convention we can apply complex Ohm's law to circuits containing capacitors and inductors, just as for resistors, once we know the reactance of a capacitor or inductor. Let's find out what these are. We have

$$V(t) = \mathcal{R}e(V_0 e^{j\omega t})$$

For a capacitor, using $I = C(dV/dt)$, we obtain

$$I(t) = -V_0 C\omega \sin\omega t = \mathcal{R}e\left(\frac{V_0 e^{j\omega t}}{-j/\omega C}\right) = \mathcal{R}e\left(\frac{V_0 e^{j\omega t}}{X_C}\right)$$

i.e., for a capacitor

$$X_C = -j/\omega C$$

X_C is the reactance of a capacitor at frequency ω. As an example a 1μF capacitor has a reactance of $-2653j$ ohms at 60Hz and a reactance of $-0.16j$ ohms at 1MHz. Its reactance at dc is infinite.

If we did a similar analysis for an inductor, we would find

$$X_L = j\omega L$$

A circuit containing only capacitors and inductors always has a purely imaginary impedance, meaning that the voltage and current are always 90° out of phase – it is purely reactive. When the circuit contains resistors, there is also a real part to the impedance. The term "reactance" in that case means the imaginary part only.

Ohm's law generalized

With these conventions for representing voltages and currents, Ohm's law takes a simple form. It reads simply

$$\mathbf{I} = \mathbf{V}/\mathbf{Z}$$
$$\mathbf{V} = \mathbf{IZ}$$

where the voltage represented by $\mathbf{V}$ is applied across a circuit of impedance $\mathbf{Z}$, giving a current represented by $\mathbf{I}$. The complex impedance of devices in series or parallel obeys the same rules as resistance:

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \cdots \quad \text{(series)}$$

$$\mathbf{Z} = \frac{1}{\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \cdots} \quad \text{(parallel)}$$

Finally, for completeness we summarize here the formulas for the impedance of resistors, capacitors, and inductors:

$\mathbf{Z}_R = R$	(resistor)
$\mathbf{Z}_C = -j/\omega C = 1/j\omega C$	(capacitor)
$\mathbf{Z}_L = j\omega L$	(inductor)

With these rules we can analyze many ac circuits by the same general methods we used in handling dc circuits, i.e., application of the series and parallel formulas and Ohm's law. Our results for circuits such as voltage dividers will look nearly the same as before. For multiply connected networks we may have to use Kirchhoff's laws, just as with dc circuits, in this case using the complex representations for V and I: The sum of the (complex) voltage drops around a closed loop is zero, and the sum of the (complex) currents into a point is zero. The latter rule implies, as with dc circuits, that the (complex) current in a series circuit is the same everywhere.

EXERCISE 1.16
Use the preceding rules for the impedance of devices in parallel and in series to derive the formulas (Section 1.12) for the capacitance of two capacitors (a) in parallel and (b) in series. Hint: In each case, let the individual capacitors have capacitances C_1 and C_2. Write down the impedance of the parallel or series combination; then equate it to the impedance of a capacitor with capacitance C. Find C.

Let's try out these techniques on the simplest circuit imaginable, an ac voltage applied across a capacitor, which we considered just previously. Then, after a brief look at power in reactive circuits (to finish laying the groundwork), we'll analyze some simple but extremely important and useful RC filter circuits.

Imagine putting a 1μF capacitor across a 110 volt (rms) 60Hz power line. What current flows? Using complex Ohm's law, we have

$$\mathbf{Z} = -j/\omega C$$

Therefore, the current is given by

$$\mathbf{I} = \mathbf{V}/\mathbf{Z}$$

The phase of the voltage is arbitrary, so let us choose $\mathbf{V} = A$, i.e. $V(t) = A\cos\omega t$, where the amplitude $A = 110\sqrt{2} \approx 156$ volts. Then

$$\mathbf{I} = j\omega CA \approx 0.059\sin\omega t$$

The resulting current has an amplitude of 59mA (41.5mA rms) and leads the voltage by 90°. This agrees with our previous calculation. Note that if we just wanted to know the magnitude of the current, and didn't care what the relative phase was, we could have avoided doing any complex algebra: If

$$\mathbf{A} = \mathbf{B}/\mathbf{C}$$

then

$$A = B/C$$

where A, B, and C are the magnitudes of the respective complex numbers; this holds for multiplication, also (see Exercise 1.17). Thus, in this case,

$$I = V/Z = \omega CV$$

This trick is often useful.

Surprisingly, there is no power dissipated by the capacitor in this example. Such activity won't increase your electric bill; you'll see why in the next section. Then we will go on to look at circuits containing resistors and capacitors with our complex Ohm's law.

EXERCISE 1.17
Show that if $\mathbf{A}=\mathbf{BC}$, then $A=BC$, where A, B, and C are magnitudes. Hint: Represent each complex number in polar form, i.e., $\mathbf{A} = Ae^{i\theta}$.

Power in reactive circuits

The instantaneous power delivered to any circuit element is always given by the product $P = VI$. However, in reactive circuits where V and I are not simply proportional, you can't just multiply them together. Funny things can happen; for instance, the sign of the product can reverse over one cycle of the ac signal. Figure 1.49 shows an example. During time intervals A and C, power is being delivered to the capacitor (albeit at a variable rate), causing it to charge up; its stored energy is increasing (power is the rate of change of energy). During intervals B and D, the power delivered to the capacitor is negative; it is discharging. The average power over a whole cycle for this example is in fact exactly zero, a statement that is always true for any purely reactive circuit element (inductors, capacitors, or any combination thereof). If you know your trigonometric integrals, the next exercise will show you how to prove this.

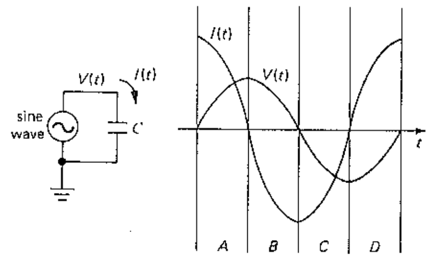


Figure 1.49. When driven by a sine wave, the current through a capacitor leads the voltage by 90°.

EXERCISE 1.18

Optional exercise: Prove that a circuit whose current is 90° out of phase with the driving voltage consumes no power, averaged over an entire cycle.

How do we find the average power consumed by an arbitrary circuit? In general, we can imagine adding up little pieces of VI product, then dividing by the elapsed time. In other words,

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt$$

where T is the time for one complete cycle. Luckily, that's almost never necessary. Instead, it is easy to show that the average power is given by

$$P = \mathcal{R}e(\mathbf{V}\mathbf{I}^*) = \mathcal{R}e(\mathbf{V}^*\mathbf{I})$$

where **V** and **I** are complex rms amplitudes.

Let's take an example. Consider the preceding circuit, with a 1 volt (rms) sine wave driving a capacitor. We'll do everything with rms amplitudes, for simplicity. We have

$$\mathbf{V} = 1$$

$$\mathbf{I} = \frac{\mathbf{V}}{-j/\omega C} = j\omega C$$

$$P = \mathcal{R}e(\mathbf{V}\mathbf{I}^*) = \mathcal{R}e(-j\omega C) = 0$$

That is, the average power is zero, as stated earlier.

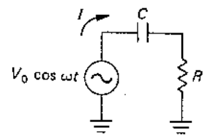


Figure 1.50

As another example, consider the circuit shown in Figure 1.50. Our calculations go like this:

$$\mathbf{Z} = R - \frac{j}{\omega C}$$

$$\mathbf{V} = V_0$$

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_0}{R - (j/\omega C)} = \frac{V_0[R + (j/\omega C)]}{R^2 + (1/\omega^2C^2)}$$

$$P = \mathcal{R}e(\mathbf{V}\mathbf{I}^*) = \frac{V_0^2 R}{R^2 + (1/\omega^2C^2)}$$

(In the third line we multiplied numerator and denominator by the complex conjugate of the denominator, in order to make the denominator real.) This is less than the product of the magnitudes of **V** and **I**. In fact, the ratio is called the *power factor*:

$$|\mathbf{V}|\,|\mathbf{I}| = \frac{V_0^2}{[R^2 + (1/\omega^2C^2)]^{1/2}}$$

$$\text{power factor} = \frac{\text{power}}{|\mathbf{V}|\,|\mathbf{I}|} = \frac{R}{[R^2 + (1/\omega^2C^2)]^{1/2}}$$

in this case. The power factor is the cosine of the phase angle between the voltage and the current, and it ranges from 0 (purely reactive circuit) to 1 (purely resistive). A power factor less than 1 indicates some component of reactive current.

EXERCISE 1.19

Show that all the average power delivered to the preceding circuit winds up in the resistor. Do this by computing the value of V_R^2/R. What is that power, in watts, for a series circuit of a 1μF capacitor and a 1.0k resistor placed across the 110 volt (rms), 60Hz power line?

Power factor is a serious matter in large-scale electrical power distribution, because reactive currents don't result in useful power being delivered to the load, but cost the power company plenty in terms of I^2R heating in the resistance of generators, transformers, and wiring. Although residential users are only billed for "real" power [$\mathcal{R}e(\mathbf{V}\mathbf{I}^*)$], the power company charges industrial users according to the power factor. This explains the capacitor yards that you see behind large factories, built to cancel the inductive reactance of industrial machinery (i.e., motors).

EXERCISE 1.20

Show that adding a series capacitor of value $C = 1/\omega^2 L$ makes the power factor equal 1.0 in a series RL circuit. Now do the same thing, but with the word "series" changed to "parallel."

Voltage dividers generalized

Our original voltage divider (Fig. 1.5) consisted of a pair of resistors in series to ground, input at the top and output at the junction. The generalization of that simple resistive divider is a similar circuit in which either or both resistors are replaced by a capacitor or inductor (or a more complicated network made from R, L, and C), as in Figure 1.51. In general, the division ratio V_{out}/V_{in} of such a divider is not constant, but depends on frequency. The analysis is straightforward:

$$\mathbf{I} = \frac{\mathbf{V}_{in}}{\mathbf{Z}_{total}}$$

$$\mathbf{Z}_{total} = \mathbf{Z}_1 + \mathbf{Z}_2$$

$$\mathbf{V}_{out} = \mathbf{I}\,\mathbf{Z}_2 = \mathbf{V}_{in}\frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}$$

Rather than worrying about this result in general, let's look at some simple, but very important, examples.

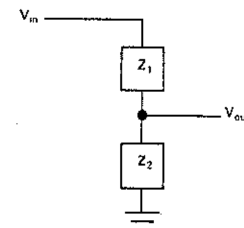


Figure 1.51. Generalized voltage divider: a pair of arbitrary impedances.

1.19 *RC* filters

By combining resistors with capacitors it is possible to make frequency-dependent voltage dividers, owing to the frequency dependence of a capacitor's impedance $\mathbf{Z}_C = -j/\omega C$. Such circuits can have the desirable property of passing signal frequencies of interest while rejecting undesired signal frequencies. In this section you will see examples of the simplest such RC filters, which we will be using frequently throughout the book. Chapter 5 and Appendix H describe filters of greater sophistication.

High-pass filters

Figure 1.52 shows a voltage divider made from a capacitor and a resistor. Complex Ohm's law gives

$$\mathbf{I} = \frac{\mathbf{V}_{in}}{\mathbf{Z}_{total}} = \frac{\mathbf{V}_{in}}{R - (j/\omega C)} = \frac{\mathbf{V}_{in}[R + (j/\omega C)]}{R^2 + 1/\omega^2C^2}$$

(For the last step, multiply top and bottom by the complex conjugate of the denominator.) So the voltage across R is just

$$\mathbf{V}_{out} = \mathbf{I}\,\mathbf{Z}_R = \mathbf{I}\,R = \frac{\mathbf{V}_{in}[R + (j/\omega C)]R}{R^2 + (1/\omega^2C^2)}$$

Most often we don't care about the phase of V_{out}, just its amplitude:

$$\mathbf{V}_{out} = (\mathbf{V}_{out}\mathbf{V}^*_{out})^{1/2}$$
$$= \frac{R}{[R^2 + (1/\omega^2C^2)]^{1/2}}V_{in}$$

Note the analogy to a resistive divider, where

$$V_{out} = \frac{R_1}{R_1 + R_2}V_{in}$$

Here the impedance of the series RC combination (Fig. 1.53) is as shown in Figure 1.54. So the "response" of this circuit, ignoring phase shifts by taking magnitudes of the complex amplitudes, is given by

$$V_{out} = \frac{R}{[R^2 + (1/\omega^2C^2)]^{1/2}}V_{in}$$
$$= \frac{2\pi fRC}{[1 + (2\pi fRC)^2]^{1/2}}V_{in}$$

and looks as shown in Figure 1.55. We could have gotten this result immediately by taking the ratio of the *magnitudes* of impedances, as in Exercise 1.17 and the example immediately preceding it; the numerator is the magnitude of the impedance of the lower leg of the divider (R), and the denominator is the magnitude of the impedance of the series combination of R and C.

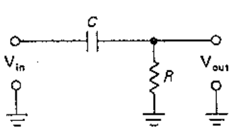


Figure 1.52. High-pass filter.

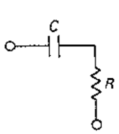


Figure 1.53

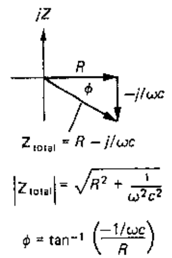


Figure 1.54

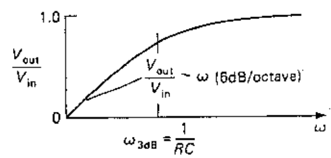


Figure 1.55. Frequency response of high-pass filter.

You can see that the output is approximately equal to the input at high frequencies (how high? $\gtrsim 1/RC$) and goes to zero at low frequencies. This is a very important result. Such a circuit is called a high-pass filter, for obvious reasons. It is very common. For instance, the input to the oscilloscope (Appendix A) can be switched to ac coupling. That's just an RC high-pass filter with the bend at about 10Hz (you would use ac coupling if you wanted to look at a small signal riding on a large dc voltage). Engineers like to refer to the −3dB "breakpoint" of a filter (or of any circuit that behaves like a filter). In the case of the simple RC high-pass filter, the −3dB breakpoint is given by

$$f_{3dB} = 1/2\pi RC$$

Note that the capacitor lets no steady current through ($f = 0$). This use as a dc *blocking capacitor* is one of its most frequent applications. Whenever you need to couple a signal from one amplifier to another, you almost invariably use a capacitor. For instance, every hi-fi audio amplifier has all its inputs capacitively coupled, because it doesn't know what dc level its input signals might be riding on. In such a coupling application you always pick R and C so that all frequencies of interest (in this case, 20Hz–20kHz) are passed without loss (attenuation).

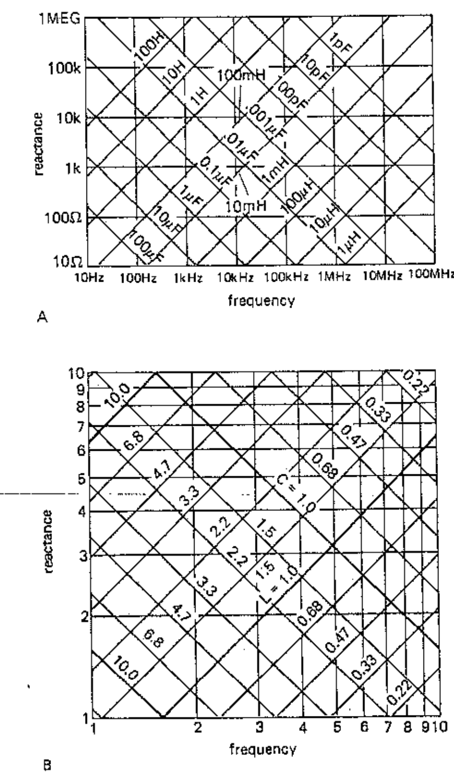


Figure 1.56. A. Reactance of inductors and capacitors versus frequency; all decades are identical, except for scale.
B. A single decade from part A expanded, with standard 20% component values shown.

You often need to know the impedance of a capacitor at a given frequency (e.g., for design of filters). Figure 1.56 provides a very useful graph covering large ranges of capacitance and frequency, giving the value of $|Z| = 1/2\pi fC$.

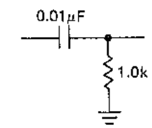


Figure 1.57

As an example, consider the filter shown in Figure 1.57. It is a high-pass filter with the 3dB point at 15.9kHz. The impedance of a load driven by it should be much larger than 1.0k in order to prevent circuit loading effects on the filter's output, and the driving source should be able to drive a 1.0k load without significant attenuation (loss of signal amplitude) in order to prevent circuit loading effects by the filter on the signal source.

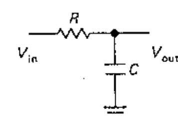


Figure 1.58. Low-pass filter.

Low-pass filters

You can get the opposite frequency behavior in a filter by interchanging R and C (Fig. 1.58). You will find

$$V_{out} = \frac{1}{(1 + \omega^2R^2C^2)^{1/2}}V_{in}$$

as seen in Figure 1.59. This is called a low-pass filter. The 3dB point is again at a frequency

$$f = 1/2\pi RC$$

Low-pass filters are quite handy in real life. For instance, a low-pass filter can be used to eliminate interference from nearby radio and television stations (550kHz–800MHz), a problem that plagues audio amplifiers and other sensitive electronic equipment.

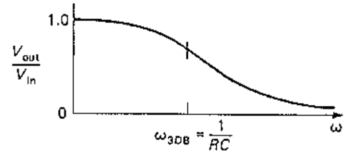


Figure 1.59. Frequency response of low-pass filter.

EXERCISE 1.21

Show that the preceding expression for the response of an RC low-pass filter is correct.

The low-pass filter's output can be viewed as a signal source in its own right. When driven by a perfect ac voltage (zero source impedance), the filter's output looks like R at low frequencies (the perfect signal source can be replaced by a short, i.e., by its small-signal source impedance, for the purpose of impedance calculations). It drops to zero impedance at high frequencies, where the capacitor dominates the output impedance. The signal driving the filter sees a load of R plus the load resistance at low frequencies, dropping to R at high frequencies.

In Figure 1.60, we've plotted the same low-pass filter response with *logarithmic* axes, which is a more usual way of doing it. You can think of the vertical axis as decibels, and the horizontal axis as octaves (or decades). On such a plot, equal distances correspond to equal ratios. We've also plotted the phase shift, using a linear vertical axis (degrees) and the same logarithmic frequency axis. This sort of plot is good for seeing the detailed response even when it is greatly attenuated (as at right); we'll see a number of such plots in Chapter 5, when we treat active filters. Note that the filter curve plotted here becomes a straight line at large attenuations, with a slope of −20dB/decade (engineers prefer to say "−6dB/octave"). Note also that the phase shift goes smoothly from 0° (at frequencies well below the breakpoint) to 90° (well above it), with a value of 45° at the −3dB point. A rule of thumb for single-section RC filters is that the phase shift is $\approx 6°$ from its asymptotic value at $0.1 f_{3dB}$ and $10 f_{3dB}$.

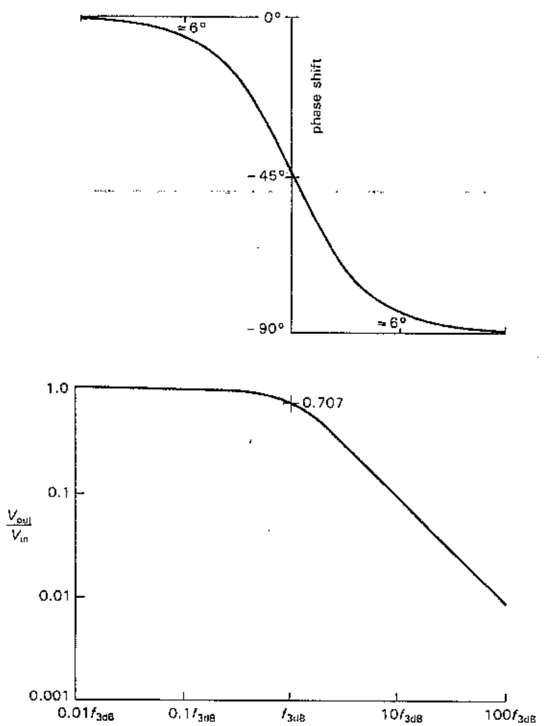


Figure 1.60. Frequency response (phase and amplitude) of low-pass filter, plotted on logarithmic axes. Note that the phase shift is 45° at the 3dB point and is within 6° of its asymptotic value for a decade of frequency change.

EXERCISE 1.22

Prove the last assertion.

An interesting question is the following: Is it possible to make a filter with some arbitrary specified amplitude response and some other specified phase response? Surprisingly, the answer is no: The demands of causality (i.e., that response must follow cause, not precede it) force a relationship between phase and amplitude response of realizable analog filters (known officially as the Kramers-Kronig relation).

RC differentiators and integrators in the frequency domain

The RC differentiator that we saw in Section 1.14 is exactly the same circuit as the high-pass filter in this section. In fact, it can be considered as either, depending on whether you're thinking of waveforms in the time domain or response in the frequency domain. We can restate the earlier time-domain condition for its proper operation ($V_{out} \ll V_{in}$) in terms of the frequency response: For the output to be small compared with the input, the signal frequency (or frequencies) must be well below the 3dB point. This is easy to check. Suppose we have the input signal

$$V_{in} = \sin \omega t$$

Then, using the equation we obtained earlier for the differentiator output,

$$V_{out} = RC \frac{d}{dt} \sin \omega t = \omega RC \cos \omega t$$

and so $V_{out} \ll V_{in}$ if $\omega RC \ll 1$, i.e., $RC \ll 1/\omega$. If the input signal contains a range of frequencies, this must hold for the highest frequencies present in the input.

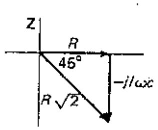
The RC integrator (Section 1.15) is the same circuit as the low-pass filter; by similar reasoning, the criterion for a good integrator is that the lowest signal frequencies must be well above the 3dB point.

Inductors versus capacitors

Inductors could be used, instead of capacitors, in combination with resistors to make low-pass (or high-pass) filters. In practice, however, you rarely see RL low- or high-pass filters. The reason is that inductors tend to be more bulky and expensive and perform less well (i.e., they depart further from the ideal) than capacitors. If you have a choice, use a capacitor. One exception to this general statement is the use of ferrite beads and chokes in high-frequency circuits. You just string a few beads here and there in the circuit; they make the wire interconnections slightly inductive, raising the impedance at very high frequencies and preventing "oscillations," without the added resistance you would get with an RC filter. An RF "choke" is an inductor, usually a few turns of wire wound on a ferrite core, used for the same purpose in RF circuits.

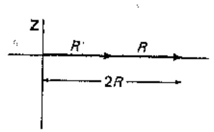
□ 1.20 Phasor diagrams

There's a nice graphic method that can be very helpful when trying to understand reactive circuits. Let's take an example, namely the fact that an RC filter attenuates 3dB at a frequency $f = 1/2\pi RC$, which we derived in Section 1.19. This is true for both high-pass and low-pass filters. It is easy to get a bit confused here, because at that frequency the reactance of the capacitor equals the resistance of the resistor; so you might at first expect 6dB attenuation. That is what you would get, for example, if you were to replace the capacitor by a resistor of the same impedance (recall that 6dB means half voltage). The confusion arises because the capacitor is reactive, but the matter is clarified by a phasor diagram (Fig. 1.61). The axes are the real (resistive) and imaginary (reactive) components of the impedance. In a series circuit like this, the axes also represent the (complex) voltage, because the current is the same everywhere. So for this circuit (think of it as an R–C voltage divider) the input voltage (applied across the series R–C pair) is proportional to the length of the hypotenuse, and the output voltage (across R only) is proportional to the length of the R leg of the triangle. The diagram represents the situation at the frequency where the magnitude of the capacitor's reactance equals R, i.e., $f = 1/2\pi RC$, and shows that the ratio of output voltage to input voltage is $1/\sqrt{2}$, i.e., -3dB.



RC filter at 3dB point

A



resistive divider: $R_1 = R_2 = R$ (−6dB)

B

Figure 1.61

The angle between the vectors gives the phase shift from input to output. At the 3dB point, for instance, the output amplitude equals the input amplitude divided by the square root of 2, and it leads by 45° in phase. This graphic method makes it easy to read off amplitude and phase relationships in RLC circuits. For example, you can use it to get the response of the high-pass filter that we previously derived algebraically.

EXERCISE 1.23

Use a phasor diagram to derive the response of an RC high-pass filter:

$$V_{\text{out}} = \frac{R}{[R^2 + (1/\omega^2 C^2)]^{1/2}} V_{\text{in}}$$

EXERCISE 1.24

At what frequency does an RC low-pass filter attenuate by 6dB (output voltage equal to half the input voltage)? What is the phase shift at that frequency?

EXERCISE 1.25

Use a phasor diagram to obtain the low-pass filter response previously derived algebraically.

In the next chapter (Section 2.08) you will see a nice example of phasor diagrams in connection with a constant-amplitude phase-shifting circuit.

1.21 "Poles" and decibels per octave

Look again at the response of the RC low-pass filter (Fig. 1.59). Far to the right of the "knee" the output amplitude is dropping proportional to $1/f$. In one octave (as in music, one octave is twice the frequency) the output amplitude will drop to half, or −6dB; so a simple RC filter has a 6dB/octave falloff. You can make filters with several RC sections; then you get 12dB/octave (two RC sections), 18dB/octave (three sections), etc. This is the usual way of describing how a filter behaves beyond the cutoff. Another popular way is to say a "3-pole filter," for instance, meaning a filter with three RC sections (or one that behaves like one). (The word "pole" derives from a method of analysis that is beyond the scope of this book and that involves complex transfer functions in the complex frequency plane, known by engineers as the "s-plane.")

A caution on multistage filters: You can't simply cascade several identical filter sections in order to get a frequency response that is the concatenation of the individual responses. The reason is that each stage will load the previous one significantly (since they're identical), changing the overall response. Remember that the response function we derived for the simple RC filters was based on a zero-impedance driving source and an infinite-impedance load. One solution is to make each successive filter section have much higher impedance than the preceding one. A better solution involves active circuits like transistor or operational amplifier (op-amp) interstage "buffers," or active filters. These subjects will be treated in Chapters 2 through 5.