Department of Physics : Inverse Problems and Imaging
Assignment 1

Due on Friday 15 April (after the mid-semester break) at 11:00am. Hand in at the start of the lecture, or before. *Please keep your assignment solutions short (but complete) and tidy.* This assignment is worth 10% of your final grade – marks *will* be deducted for untidy work.

1. Consider trying to reconstruct a bounded function $f(t)$ from M measurements of its Laplace transform made for positive-real s, i.e., from

$$d_l = \int_0^\infty e^{-s_l t} f(t)\, dt, \qquad l = 1, 2, \ldots, M,$$

where all s_l are positive and real. We would like to choose the values of the $\{s_l\}$ to make the inverse problem as well-behaved as possible. By investigating the forward problems for the cases:

   (a) $s_l = l/M$, $M = 10, 30, 100$

   (b) $s_l = 2l/M$, $M = 10, 30, 100$

   (c) $s_l = 10l/M$, $M = 10, 30, 100$

indicate which set of measurements would be best in the sense of the largest number of singular values above 10^{-6} of the maximum singular value. (This measure is relevant when measurement error is 1 part in 10^6, i.e., SNR=10^6.)

2. (a) Show that for a unitary matrix U, $\|Uv\|^2 = \|v\|^2$ for any vector v.

   (b) The finite Fourier transform of a (possibly complex-valued) sequence $\{x[k]\}_{k=0}^{N-1}$ is a sequence $\{X[r]\}_{r=0}^{N-1}$defined by,

$$X[r] = \sum_{k=0}^{N-1} x[k] \exp\left(-j\frac{2\pi rk}{N}\right)$$

   and the inverse finite Fourier transform is given by

$$x[k] = \frac{1}{N}\sum_{r=0}^{N-1} X[r] \exp\left(+j\frac{2\pi rk}{N}\right).$$

   Use the result of the previous part to show that

$$\sum_{k=0}^{N-1} |x[k]|^2 = \frac{1}{N}\sum_{r=0}^{N-1} |X[r]|^2 .$$

   (c) In the deblurring problem discussed in lectures, after appropriate zero-padding the data $d[k]$ are related to the image $f[k]$ and the point-spread function $h[k]$ via

$$d[k] = (f \circledast h)[k] + n[k]$$

   where $\circledast$ denotes circular convolution and $n[k]$ is the noise. Consider the following method for obtaining a reconstruction $\hat{f}[k]$ of the image. We choose $\hat{f}[k]$ so as to minimize the sum of squares

$$\sum_{k=0}^{N-1} \left|\hat{f}[k]\right|^2$$

subject to the constraint that the data misfit is equal to some specified value C_0 (which is set by the expected amount of noise), i.e.,

$$\sum_{k=0}^{N-1} \left| d[k] - \left(\hat{f} \circledast h \right)[k] \right|^2 = C_0$$

By introducing a Lagrange multiplier in order to impose the constraint, show that the finite Fourier transform $\hat{F}[r]$ of the reconstructed image is given by

$$\hat{F}[r] = \frac{D[r]\, H[r]^*}{\lambda^2 + |H[r]|^2},$$

where λ^2 is a quantity related to the Lagrange multiplier.

(d) Show that the reconstruction formula derived in (c) is exactly the formula involving 'filter factors' derived in class. (Your work in Q5 is Assignment 1 may come in handy here.)

3. Apply the result of the previous question to perform deblurring and hence sharpen up a photograph of jupiter. On the 445 website (`https://coursesupport.physics.otago.ac.nz/wiki/pmwiki.php/ELEC445/HomePage`) you will find the following two files:

   (a) `jupiter1.tif` which contains a photograph of Jupiter taken in the methane band (780nm) on a grid of size 256×256 pixels, each takes an integer value from 0 to 255. In the upper right-hand portion of the photograph is one of the Galilean satellites. This satellite is effectively a point source, and so we can use that region of the photograph as the point-spread function.

   (b) `jupiter.m` contains Matlab code to read in the photograph, display it, and extract a 32×32 window of points around the satellite as the point spread function. Note that the function `double` is used to convert the 16 bit integers of the photograph to the "double precision" real numbers used in Matlab.

   Write a programme to carry out deconvolution of the photograph using the point spread function, in an attempt to improve the resolution of the image. Being a two-dimensional image, we need to use the two-dimensional Fourier transform functions `fft2` and `ifft2` in Matlab. Zero-pad the point spread function to size 256×256 (check the help text for `fft2`) and assume that the forward map is a circular convolution. Make a printout of the original blurred image. Obtain and plot the singular-values of the forward map sorted in decreasing order (don't use MatLab's `svd` command unless you want to wait a long time). Attempt to carry out the deconvolution by direct division in the Fourier domain and make a printout of the result. Next reconstruct the image using Tikhonov regularisation. Show a picture of the L-curve and choose a suitable value of the regularizing parameter λ to display your 'best' reconstruction.

   In your deconvolved picture, you should notice the bands around the planet due to convection of the atmosphere. Impacts from the recent (July 1994) collision of the Shoemaker-Levy 9 comet are visible in the upper part of the disc and the Red Spot is on the limb of the planet on the right side. The bright spot in the lower part of the image is the transit of one of the other Galilean satellites.

   Challenge question (for interest only): You will see in your deblurred image that there are some non-physical ripples around the deconvolved Galilean satellite. Can you improve the reconstruction by fiddling with the point-spread function that you use?