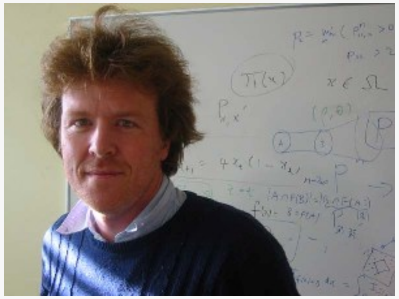
Mid- and high-level representations for inverse problems in PDEs

Colin Fox

fox@physics.otago.ac.nz

Ville Kohlemainen (Kuopio), Geoff Nicholls (Oxford UK)

Markus Neumayer (Graz), Daniel Watzenig (Graz)

Representations

Observation space is determined (finite set of numbers on a computer)

How to represent the unknown x is *always* a modelling choice

Spatially-distributed parameters often modelled using stochastic models from spatial statistics, pattern theory, stochastic geometry :

Hurn Husby & Rue (2003) classified representations/priors as

- Low level: pixel based, linear space, often GMRF, can impose local properties
- Mid level: capture some global features, often good for geometric information, e.g. boundaries/areas
- High level: objects modelled directly, good for counting number of objects

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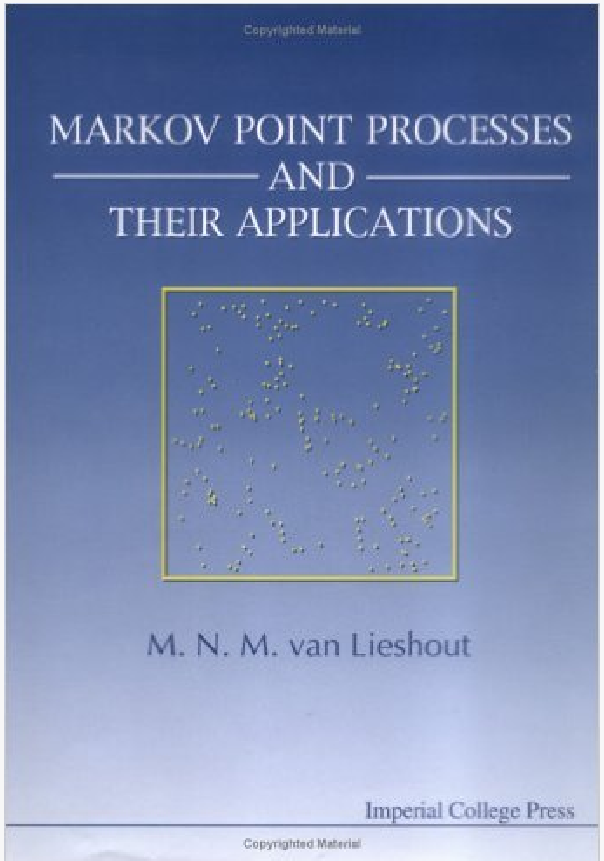
Representation of knowledge in complex systems, Grenander & Miller JRSSB 1994

3 books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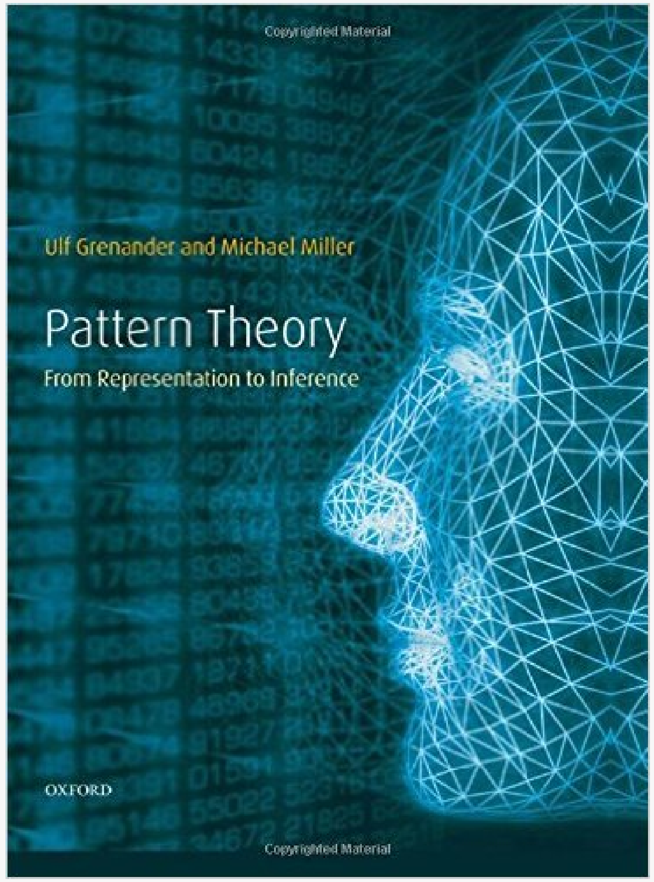
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Ulf Grenander and Michael Miller
Pattern Theory
From Representation to Inference
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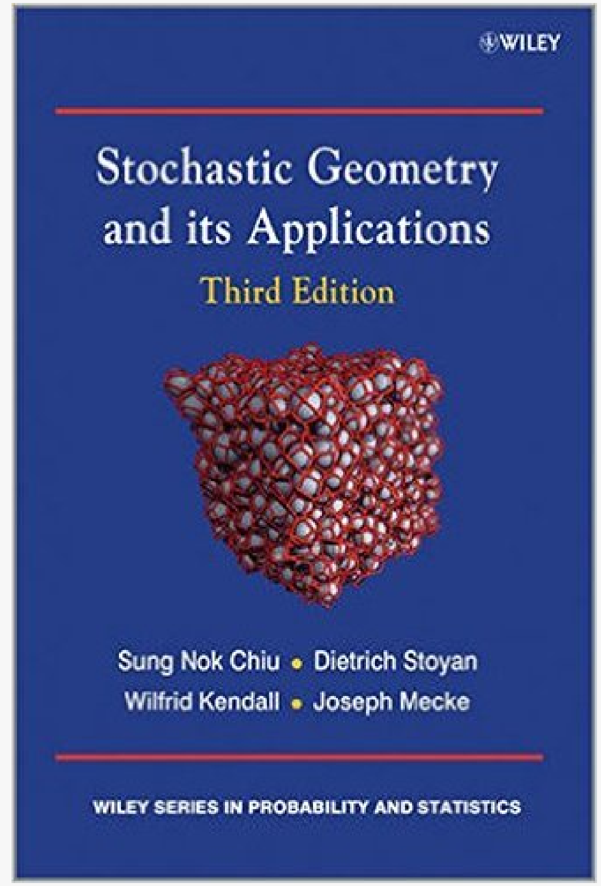
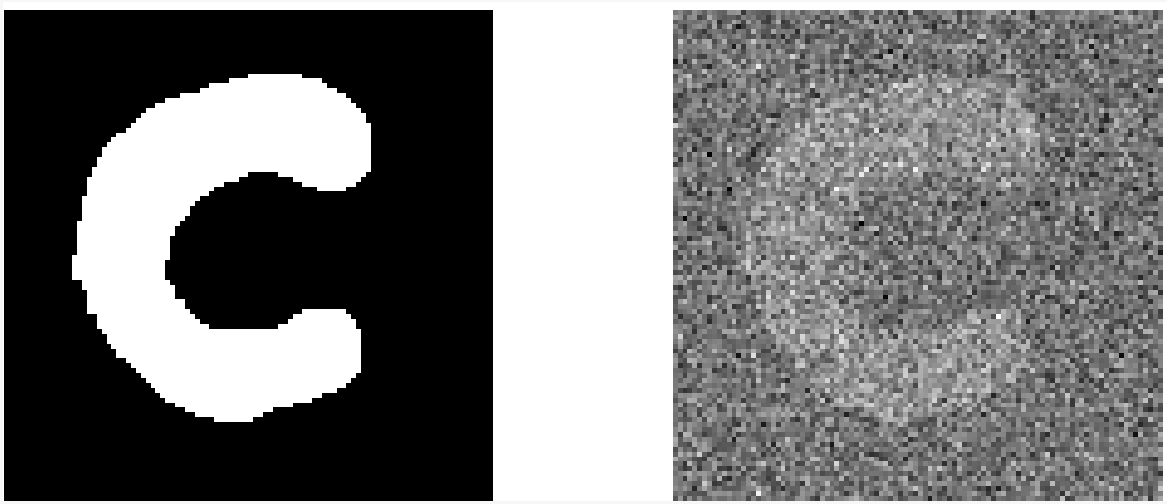

WILEY
Stochastic Geometry
and its Applications
Third Edition
Sung Nok Chiu • Dietrich Stoyan
Wilfrid Kendall • Joseph Mecke
WILEY SERIES IN PROBABILITY AND STATISTICS

What questions are we trying to answer?

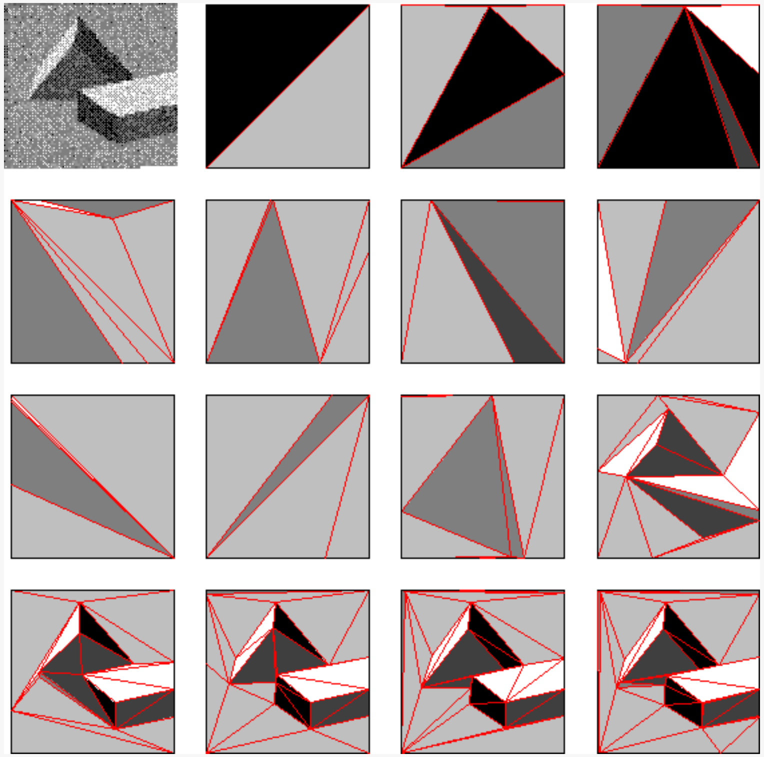


- "best" image
- How many blobs (when segmented into black and white)?
- What is the area of the blob ?
- Genus of the blob? ('C' or 'O')

A representation should make it easy to calculate information or quantities of interest.

If you want to know where the boundary is, then represent the boundary explicitly!

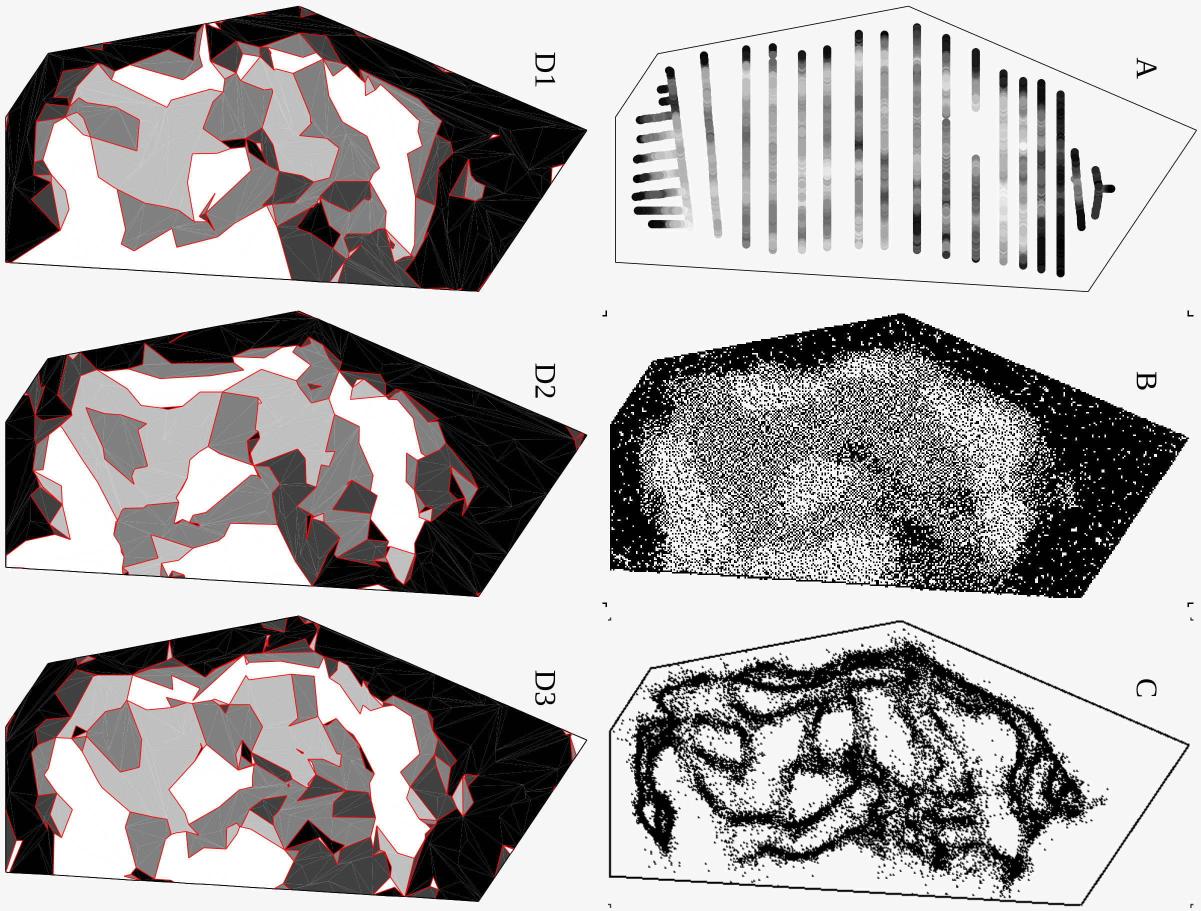
Coloured Continuum Triangulation



$$X = \bigcup_{i=0}^{\infty} \{[0,1] \times [0,1]\}^i \text{, coloured}$$

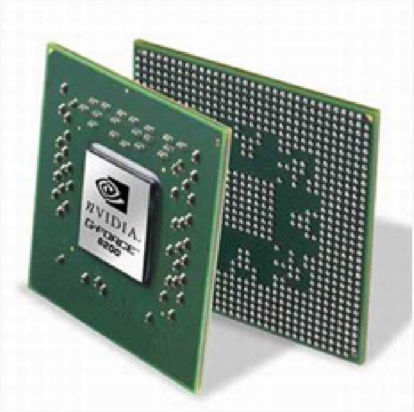
Geoff Nicholls, *Bayesian image analysis with Markov chain Monte Carlo and colored continuum triangulation models* JRSSB **60**:3 643-659 (1998)

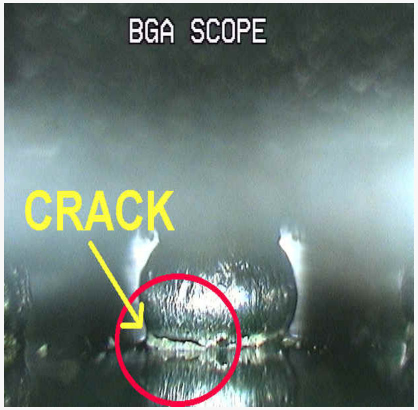
Neolithic hill fort (Maori pa)



A) data, 1746 resistivity readings, (B) posterior mean resistivity, (C) posterior edge length density, (D1-3) samples from posterior

Automated inspection of BGAs by limited-angle X-ray

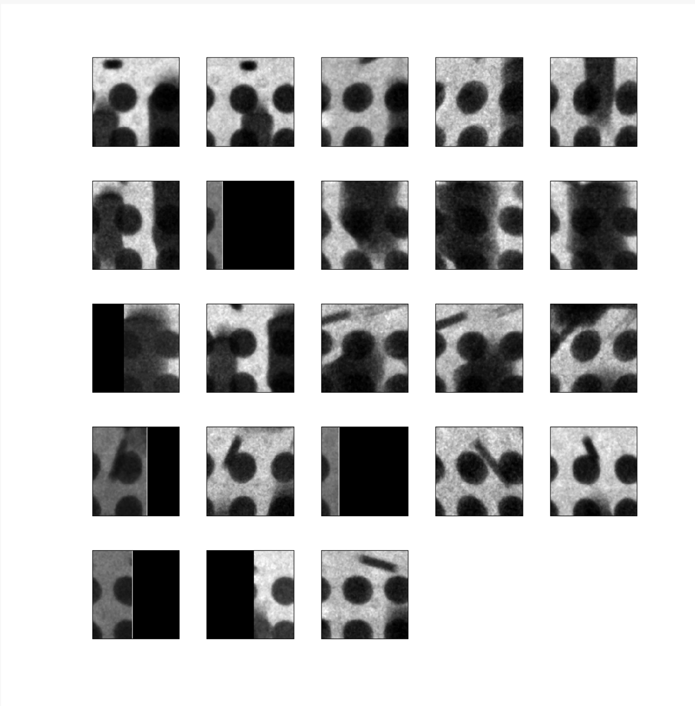


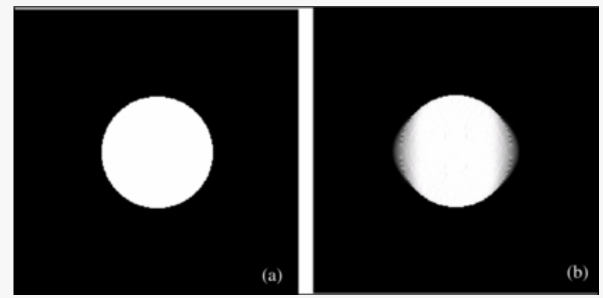


Low-level representation gives ‘coneheads’

Standard processing is:

- Produce pixel/voxel image



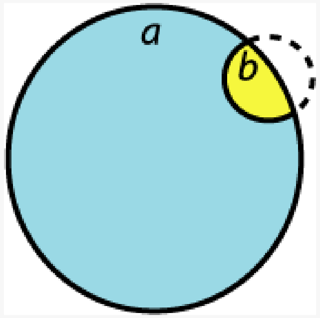


- Classify image

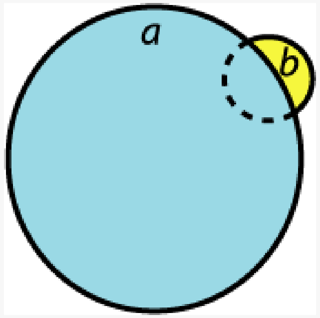
$\geq 5\%$ misclassification is no use for consumer electronics

CSG representation

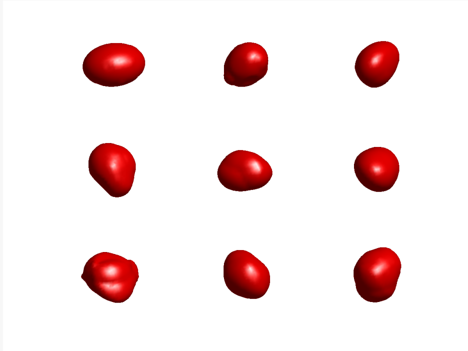
a contains b

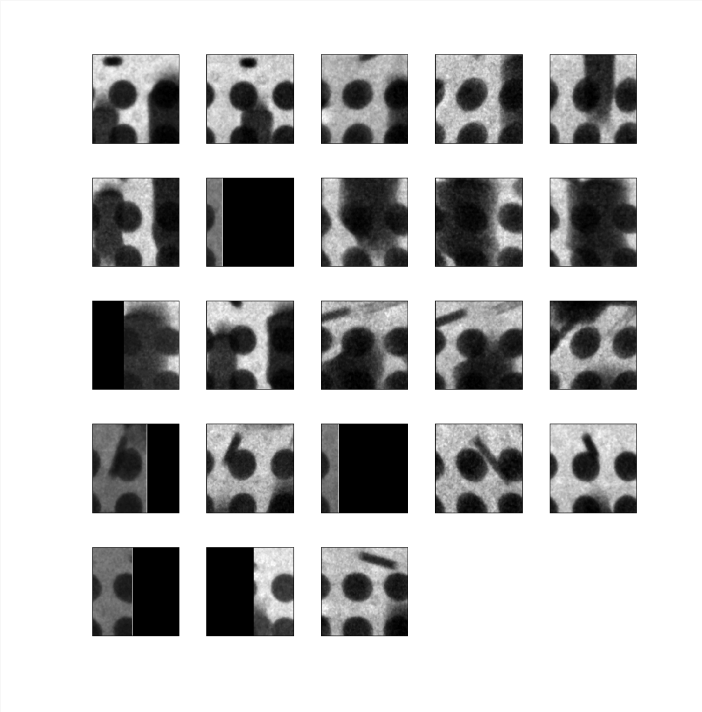


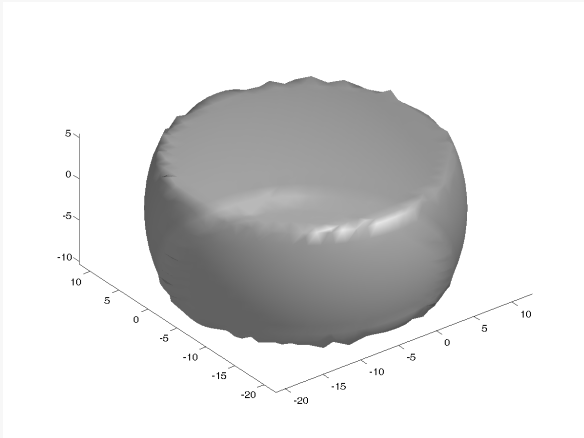
a excludes b



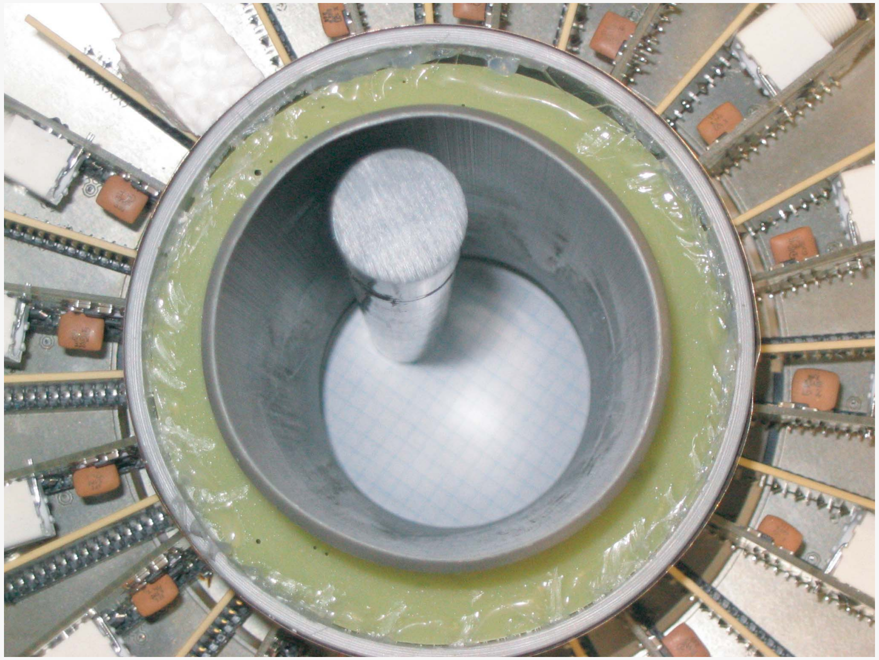
primitives







Electrical capacitance tomography

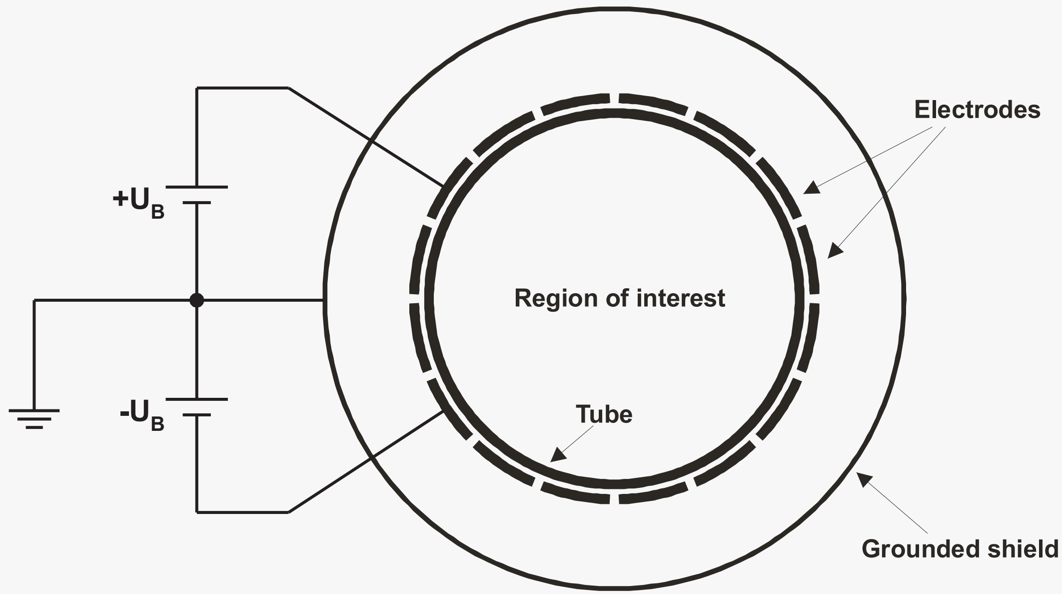


- Measure inter-electrode capacitances (1 fF to 5 pF)

$$\mathbf{q} = C\mathbf{v}$$

- Non-invasively image permittivity ε
- Primarily interested in (2-dim) area of inclusion

ECT measurement system



Assert N_M potential vectors $\mathbf{v}^m = \left\{v_1^m, v_2^m, \ldots, v_{N_\mathsf{E}}^m\right\}^\mathsf{T}$, for $m = 1, 2, \ldots, N_\mathsf{M}$

Resulting potential fields denoted u^m

Measure vector of (displacement) charges is $\mathbf{q}^m = \left\{q_1^m, q_2^m, \ldots, q_{N_\mathsf{E}}^m\right\}^\mathsf{T}$

$\mathbf{q}^m$ is a *linear* function of $\mathbf{v}^m$, hence

$$\mathbf{q} = C\mathbf{v}$$

where C is the $N_\mathsf{E} \times N_\mathsf{E}$ matrix of trans-capacitances.

Forward map G

ECT

$$\nabla \cdot (\varepsilon \nabla u) = 0 \qquad \text{in } \Omega \cup \Omega_{\mathsf{E}}$$

$$u|_{\partial\Omega_k} = v_k \qquad k = 1, 2, \ldots, N_{\mathsf{E}}, S$$

Measured charge related to fields by

$$q_k = \int_{\partial\Omega_k} \varepsilon \nabla u \cdot \mathbf{n}\, dl, \qquad k = 1, 2, \ldots, N_{\mathsf{E}}$$

EIT

$$\nabla \cdot \sigma \nabla u = 0 \qquad \text{in } \Omega$$

$$\int_{e_l} \sigma \frac{\partial u}{\partial n} dS = I_l$$

$$\sigma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \setminus \bigcup_l e_l} = 0$$

$$\left(u + z_l \sigma \frac{\partial u}{\partial n} \right)\bigg|_{e_l} = U_l$$

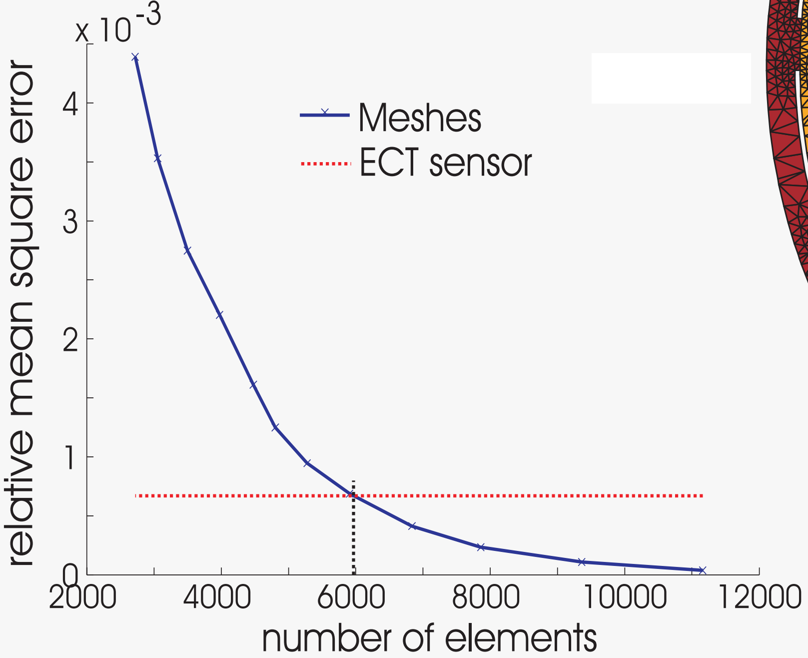
Data simulation requires $N_{\mathsf{M}} \sim 16$ solves of the Dirichlet (Neumann) BVP.

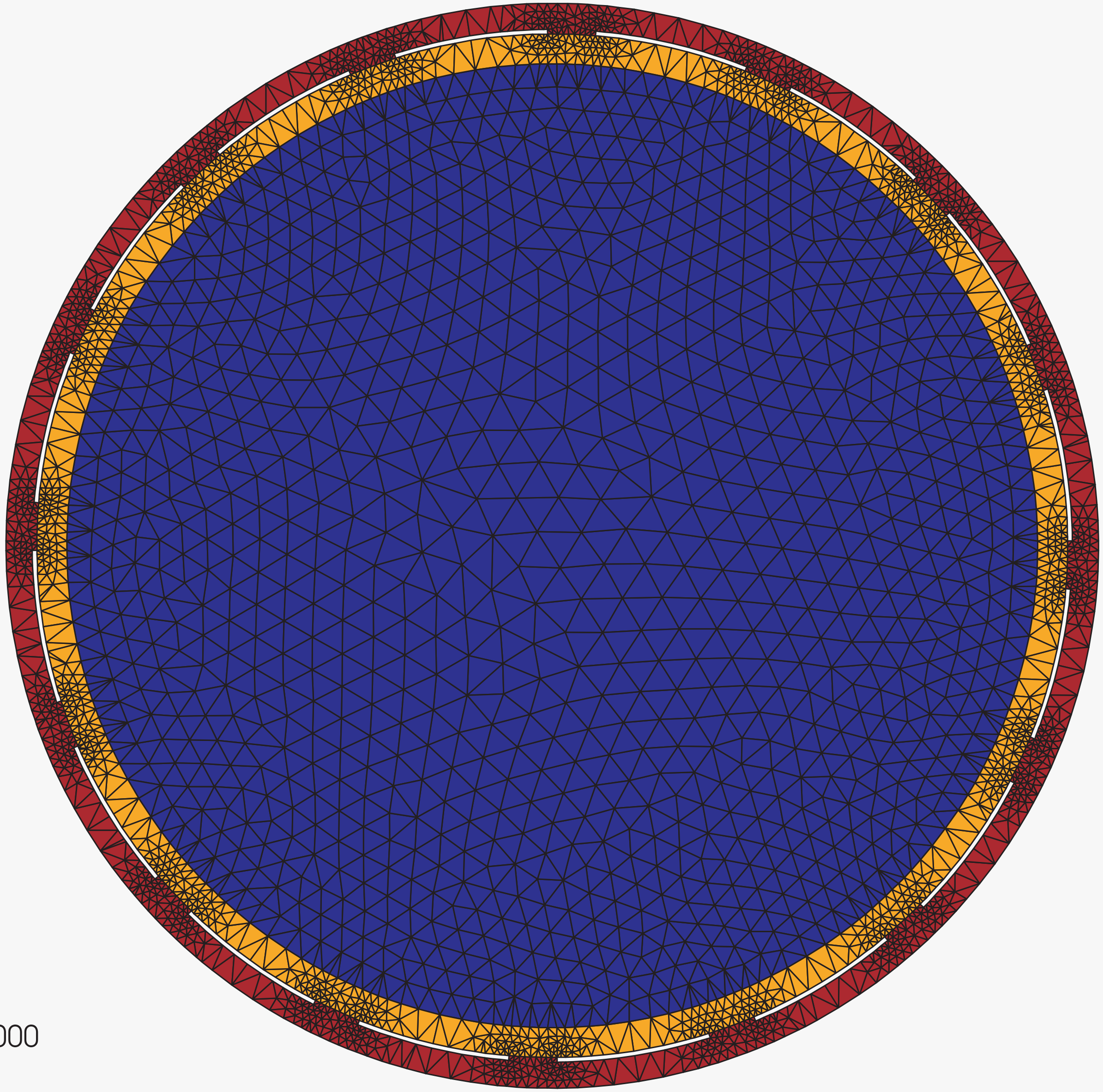
SNR of 1:1000 provides 105 measurements + 5 per factor of 10 (further measurements give $\sqrt{n}$ noise improvement). Correlation = 1-0.

Big names (Ohm, Kirchhoff, Laplace, Maxwell), but the biggest source of error!

$$\pi(\varepsilon \mid \mathbf{q}) \approx \pi_{\mathrm{n}}(\mathbf{q} - G(\varepsilon))\pi_{\mathrm{pr}}(\varepsilon)$$

FEM Mesh for ECT





Gaussian smoothness prior

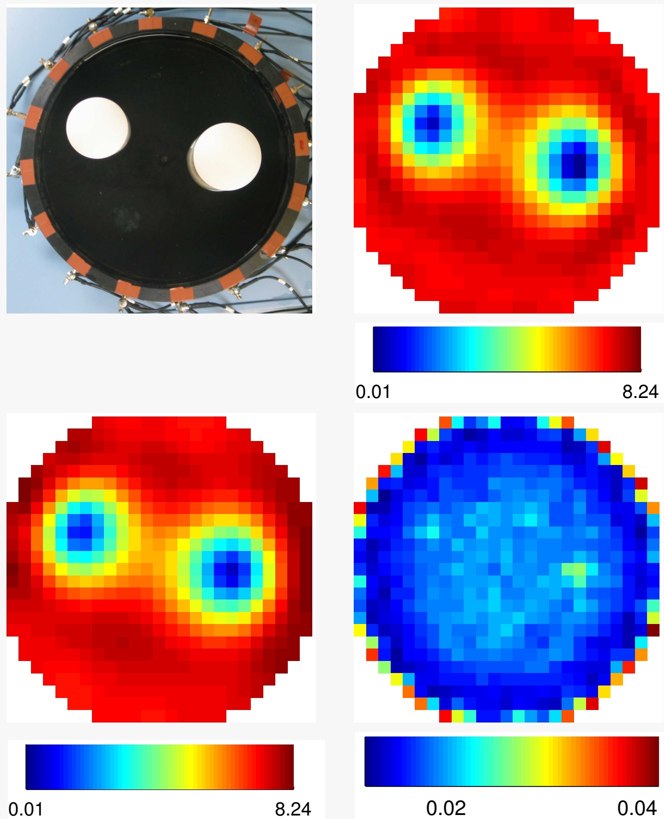


Figure 1: Results with the Gaussian smoothness MRF-prior. Top left: Photograph of the measurement setup. Top right: Maximum a posteriori estimate σ_{MAP} by the Gauss-Newton optimization algorithm. Bottom left and right: Posterior mean σ_{CM} and variance based on the MCMC simulation.

Material type prior

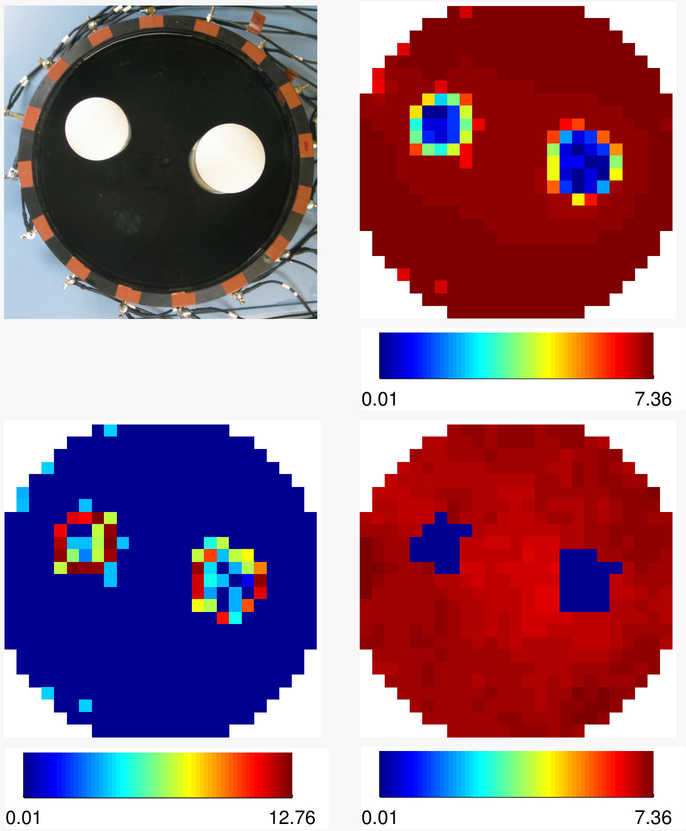
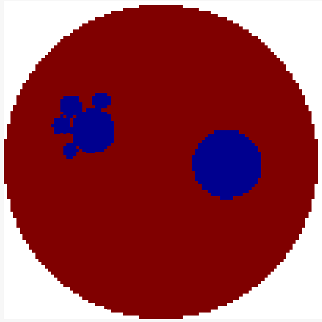


Figure 3: Results with the Material type MRF-prior. Top left: Photograph of the measurement setup. Top right: Posterior mean for the conductivity. Bottom left: Posterior variance of the conductivity. Bottom right: One sample from the posterior.

Nicholls F 1998

Circular inclusions prior



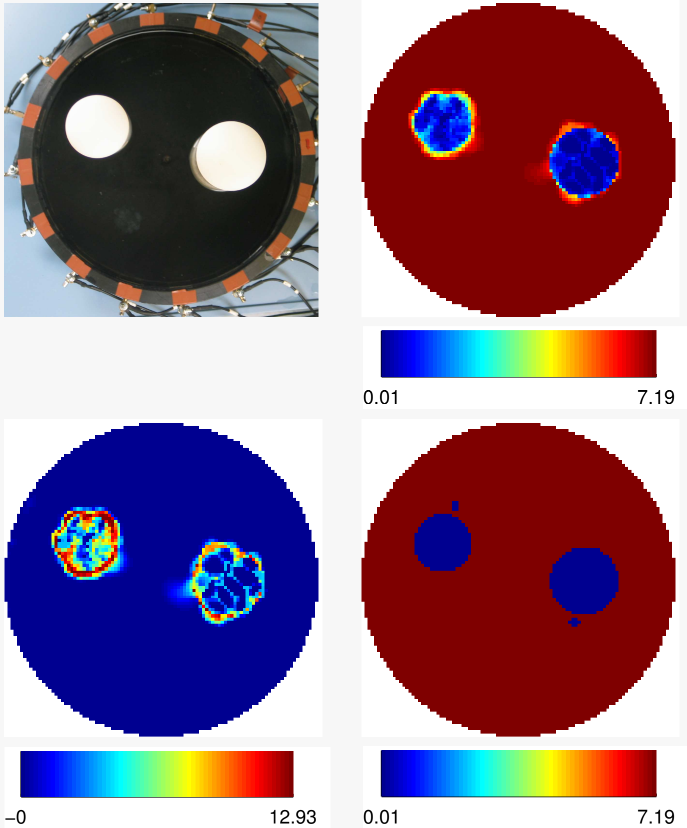


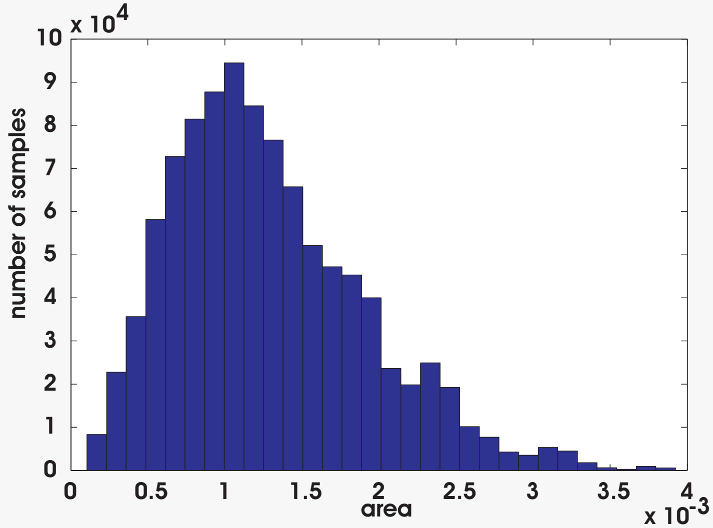
Figure 5: Results with the circle prior. Top left: Photograph of the measurement setup. Top right: Posterior mean for the conductivity. Bottom left: Posterior variance of the conductivity. Bottom right: Sample from the posterior.

Represent boundary by implicit RBF (or polygon)

Represent boundary by N point implicit RBF x

Naive prior uniform in node position: $\pi_{\text{pr}}(x) = I(\text{allowable contour})$

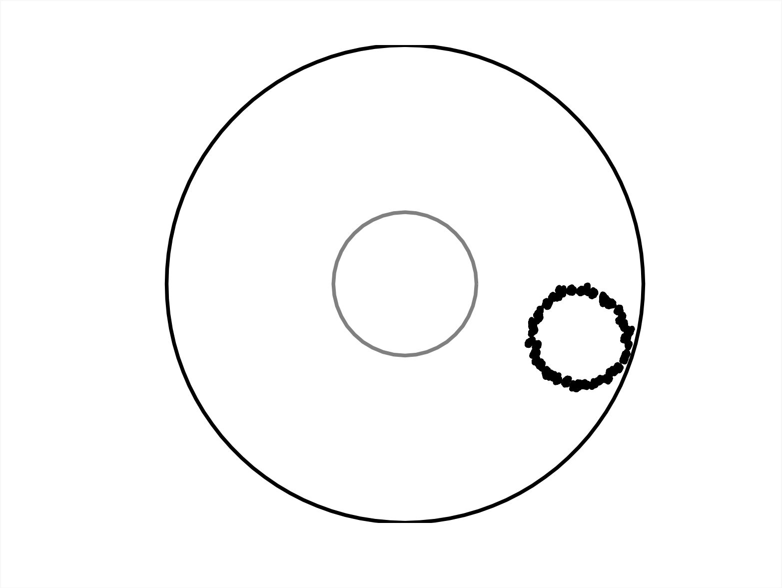
For large area $\pi_{\text{pr}}(\text{area}) \propto (\text{area})^{-1/2}$

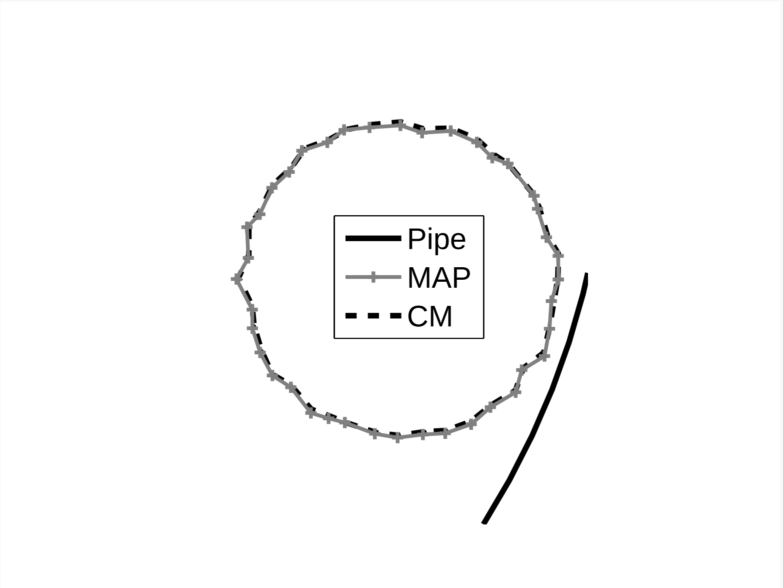


Specify a prior explicitly in terms of area $\Gamma(x)$ and circumference $c(x)$

$$\pi(x) \propto \exp\left\{-\frac{1}{2\sigma_{\text{pr}}^2}\left(\frac{c(x)}{2\sqrt{\Gamma(x)\pi}} - 1\right)\right\} I\left(x\right)$$

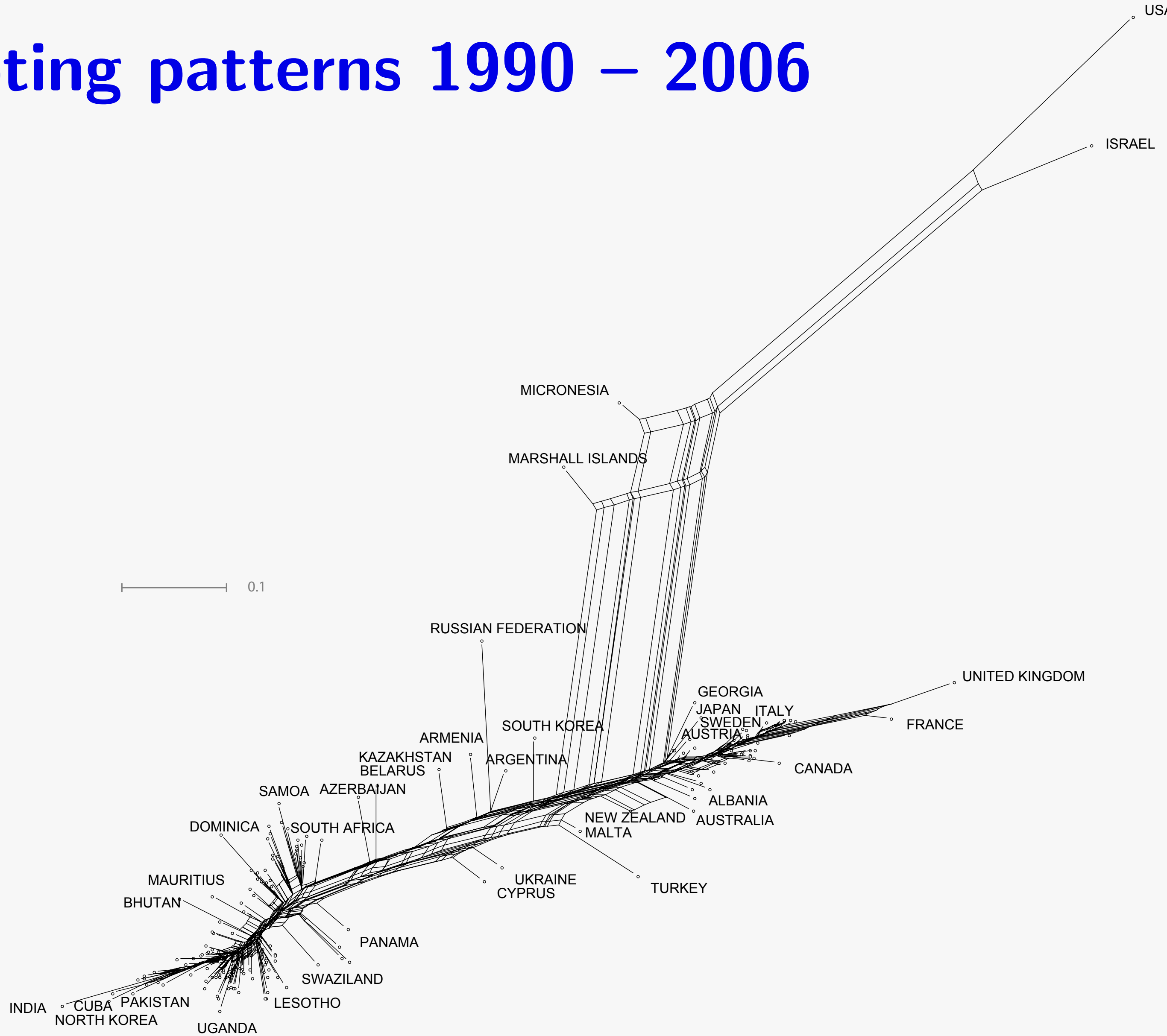
Posterior estimates (measured data)





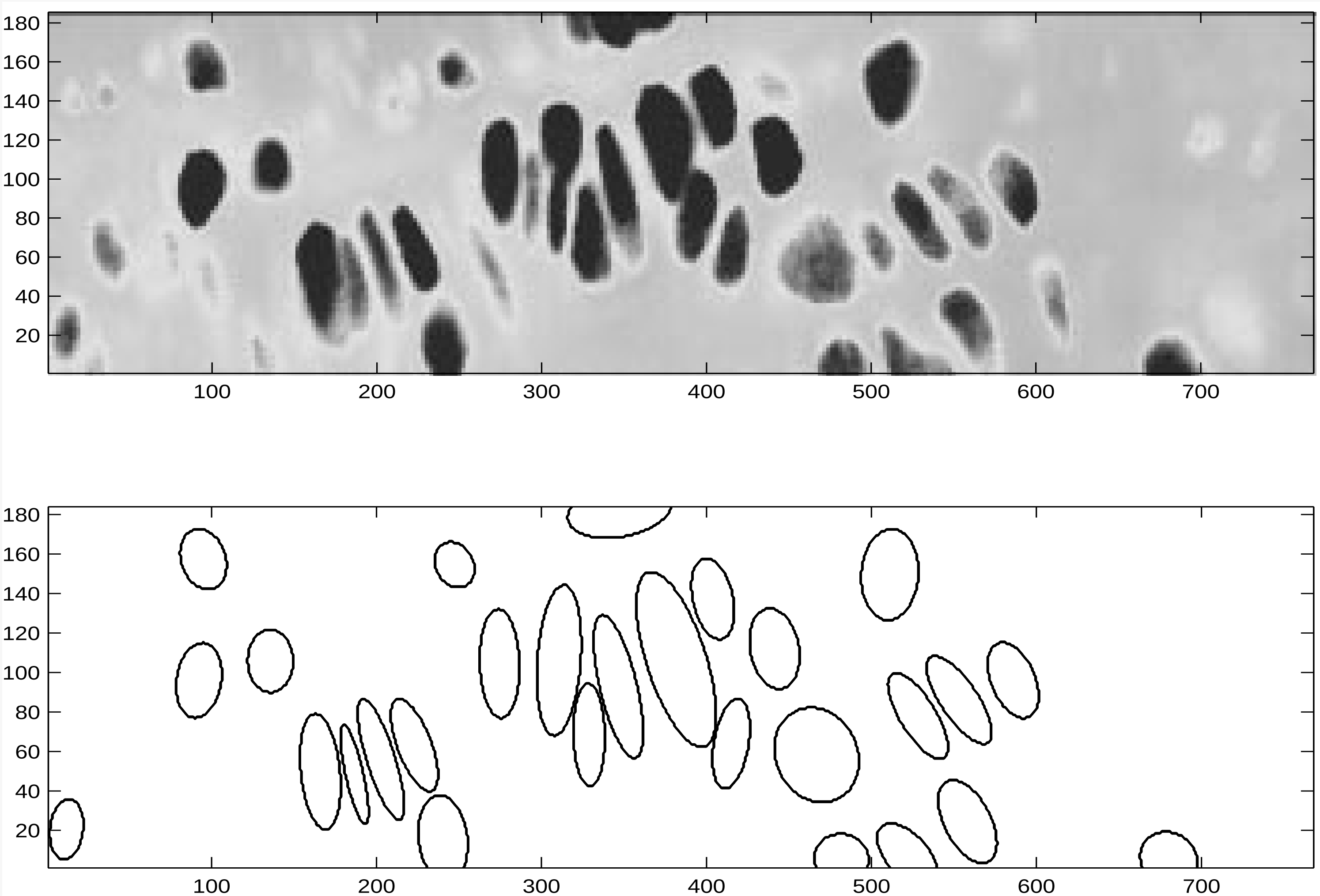
Quantities	true values	mean	standard deviation	IACT
x-coordinate of center [m]	–	3.71×10^{-2}	2.32×10^{-5}	5.89×10^{2}
y-coordinate of center [m]	–	-1.14×10^{-2}	3.02×10^{-5}	4.65×10^{2}
Area Γ [m^2]	3.14×10^{-4}	3.13×10^{-4}	6.88×10^{-6}	1.10×10^{3}
Circumference c [m]	6.28×10^{-2}	6.24×10^{-2}	1.57×10^{-4}	1.88×10^{3}
Log-likelihood	–	-46.10	1.72×10^{-1}	3.99×10^{2}

U.N. voting patterns 1990 – 2006



thanks to David Bryant

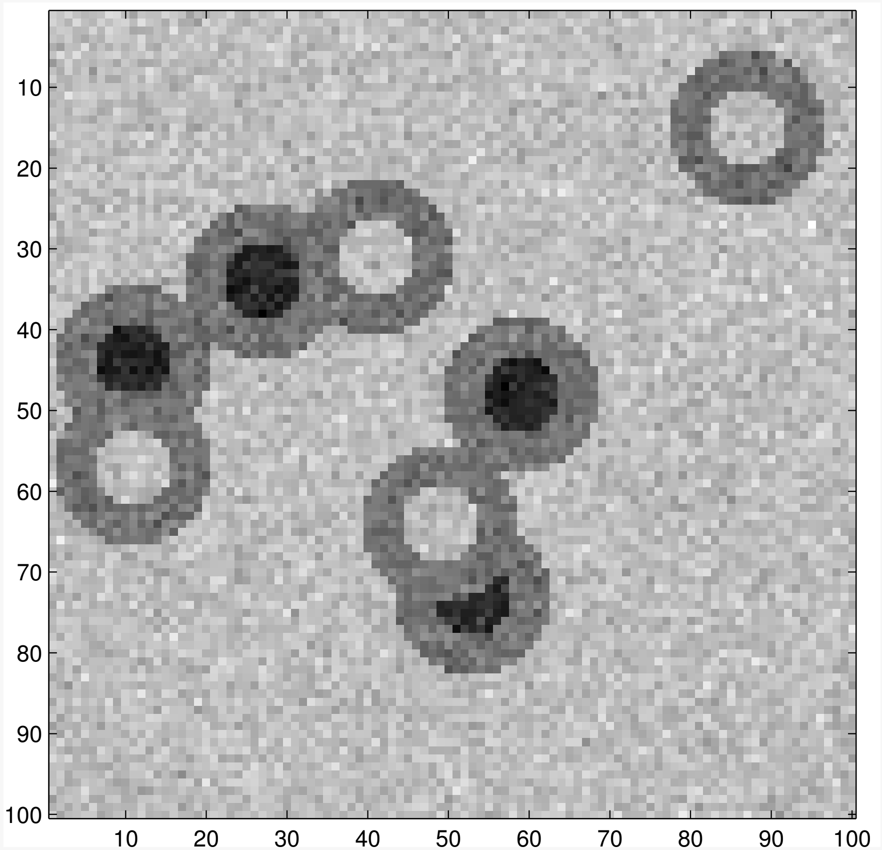
Marked Point Process



Fahimah Al-Awadhi, Christopher Jennison, Merrilee Hurn (Applied Statistics 53(1):31-49 2004)

Computing 3

Independent pixel-wise observations, with Gaussian noise, of ‘cells’.



Count the number of good (black inside) and bad (white inside) cells in this (synthetic) image

... using an MCMC with a marked point process representation.