Mid- and high-level representations for inverse problems in PDEs

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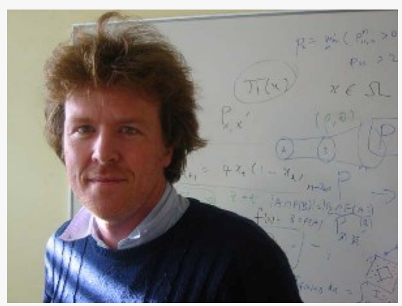

Ville Kohlemainen (Kuopio), Geoff Nicholls (Oxford UK)
Markus Neumayer (Graz), Daniel Watzenig (Graz)

Representations

Observation space is determined (finite set of numbers on a computer)

How to represent the unknown x is *always* a modelling choice

Spatially-distributed parameters often modelled using stochastic models from spatial statistics, pattern theory, stochastic geometry :

Hurn Husby & Rue (2003) classified representations/priors as

- Low level: pixel based, linear space, often GMRF, can impose local properties
- Mid level: capture some global features, often good for geometric information, e.g. boundaries/areas
- High level: objects modelled directly, good for counting number of objects

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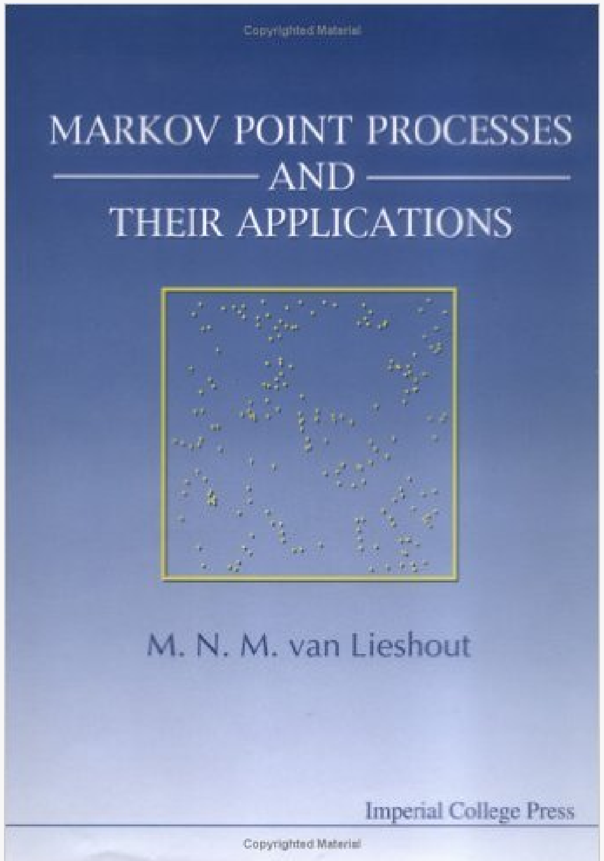
Representation of knowledge in complex systems, Grenander & Miller JRSSB 1994

3 books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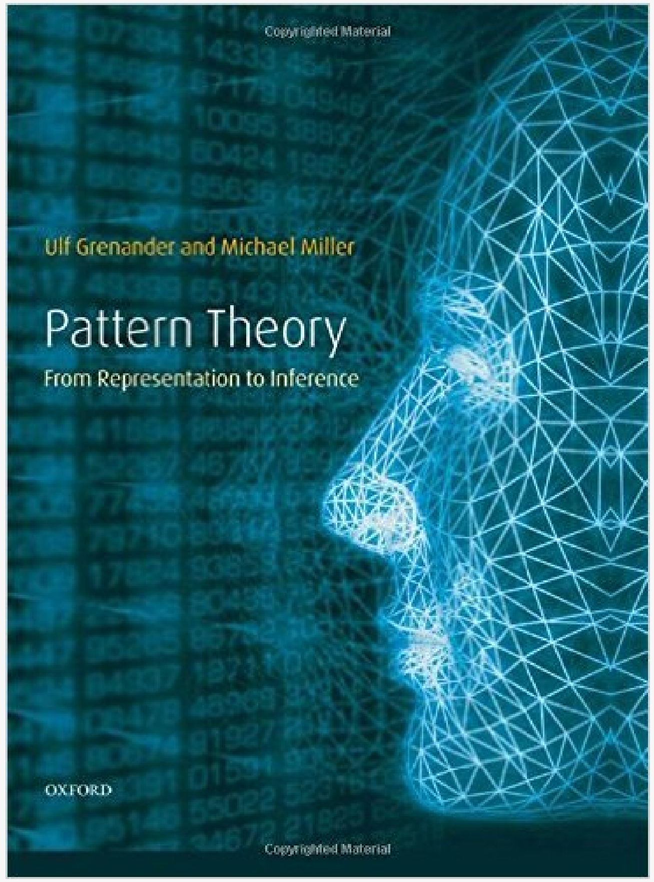
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Ulf Grenander and Michael Miller
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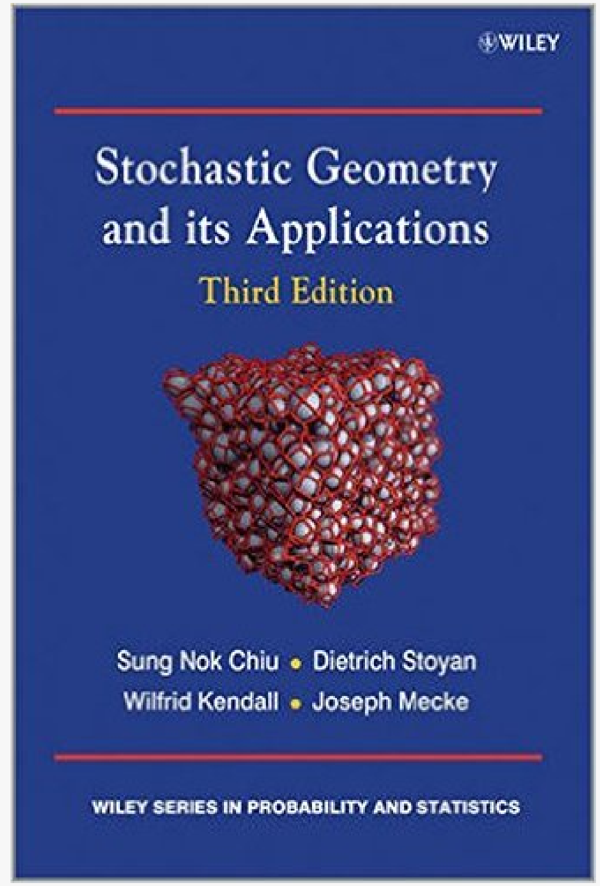
WILEY
Stochastic Geometry
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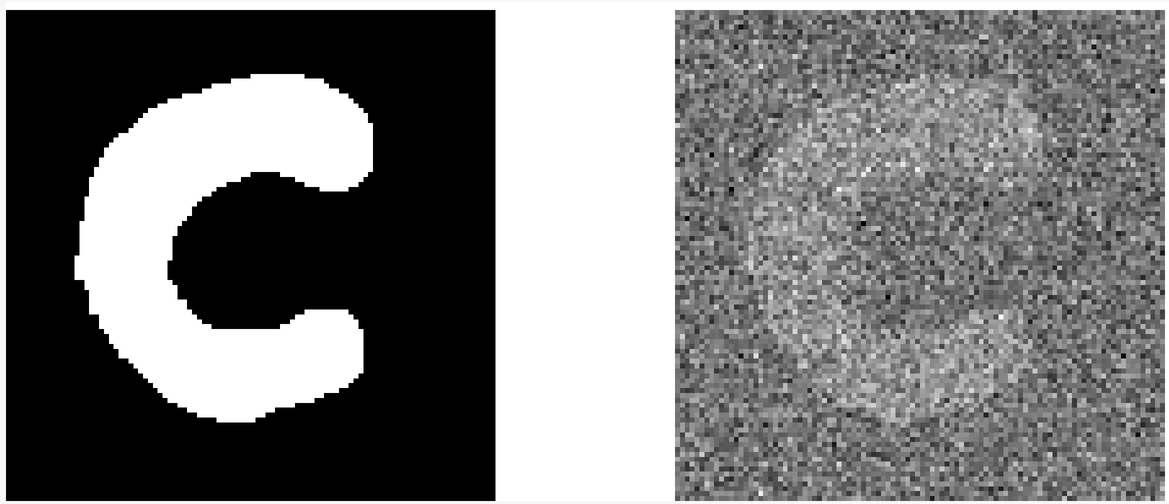
Third Edition
Sung Nok Chiu • Dietrich Stoyan
Wilfrid Kendall • Joseph Mecke
WILEY SERIES IN PROBABILITY AND STATISTICS

What questions are we trying to answer?

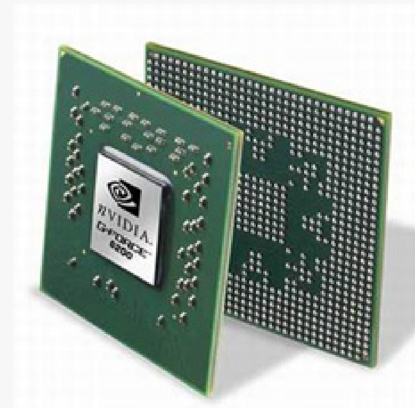


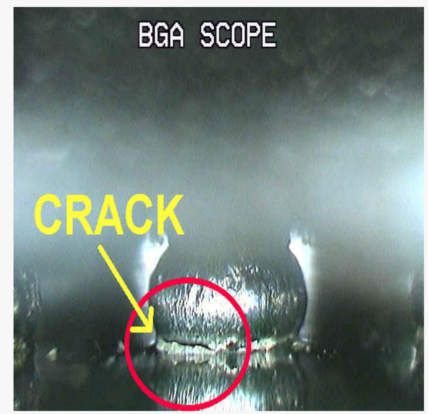
- "best" image
- How many blobs (when segmented into black and white)?
- What is the area of the blob ?
- Genus of the blob? ('C' or 'O')

A representation should make it easy to calculate information or quantities of interest.

If you want to know where the boundary is, then represent the boundary explicitly!

Automated inspection of BGAs by limited-angle X-ray

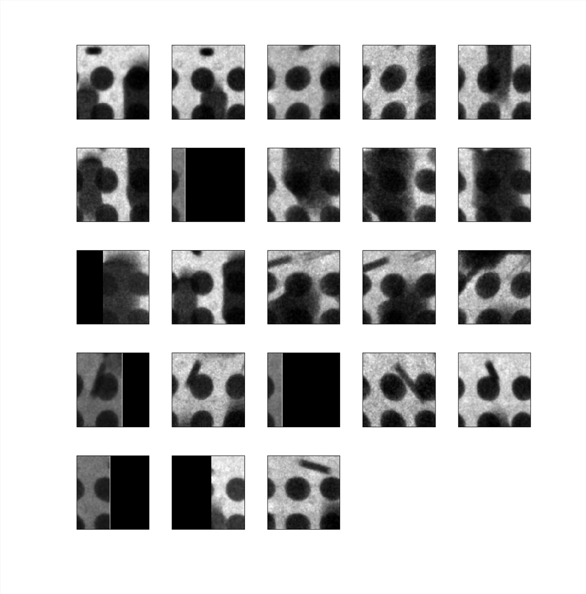


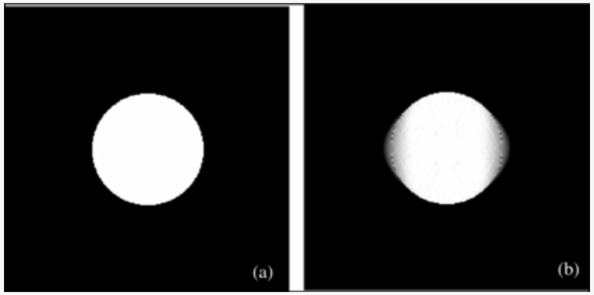


Low-level representation gives ‘coneheads’

Standard processing is:

- Produce pixel/voxel image

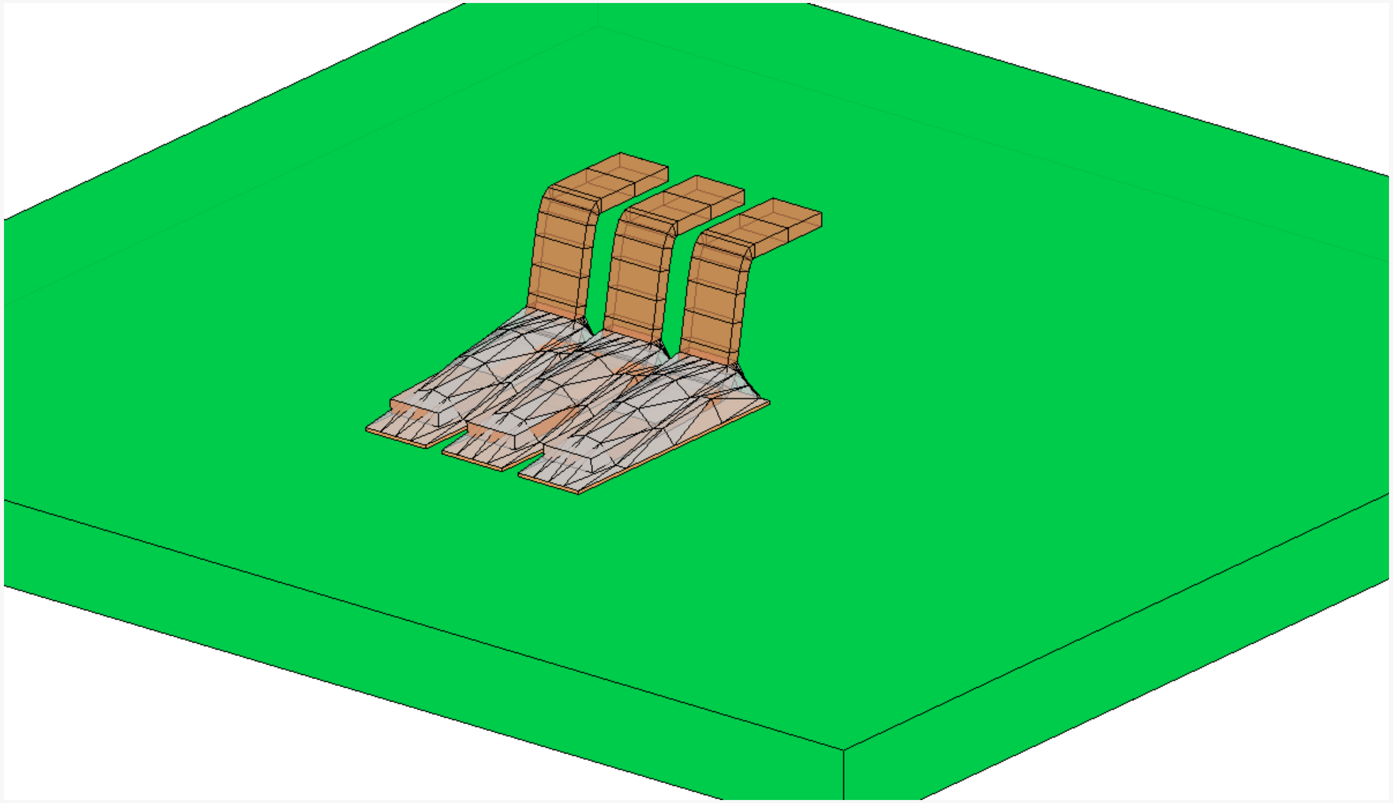




- Classify image

$\geq 4\%$ misclassification unprofitable for consumer electronics (this gives $\approx 20\%$)

Mid-level representation (surface)

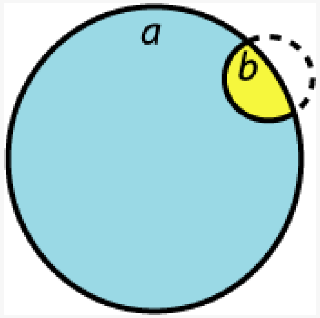


Inverse problem ill-posed with low-level representation (about 15 times too few measurements)

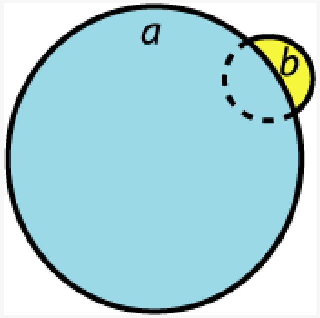
Surface representation shows actually about 5 times more than enough measurements

CSG representation

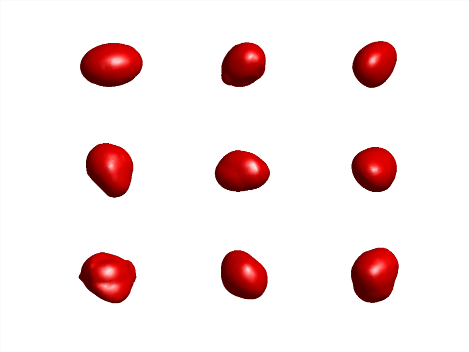
a **contains** *b*

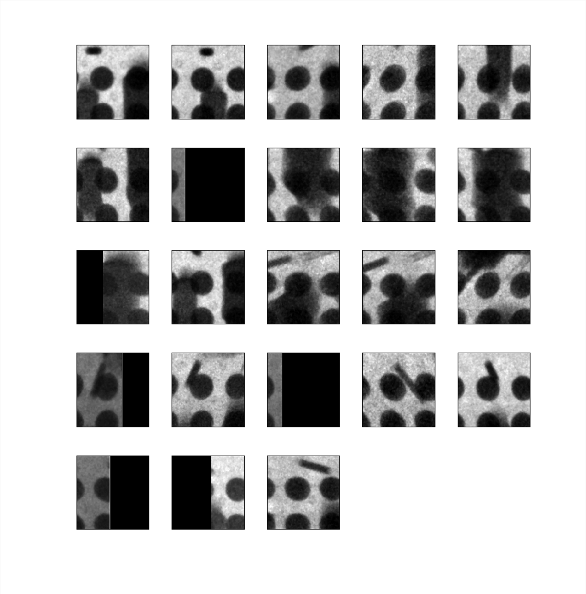


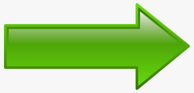
a **excludes** *b*

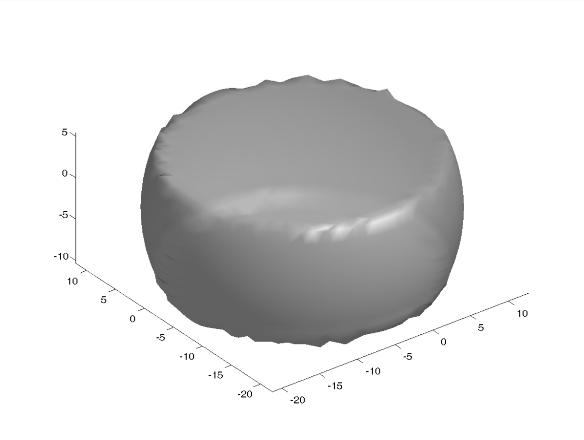


primitives



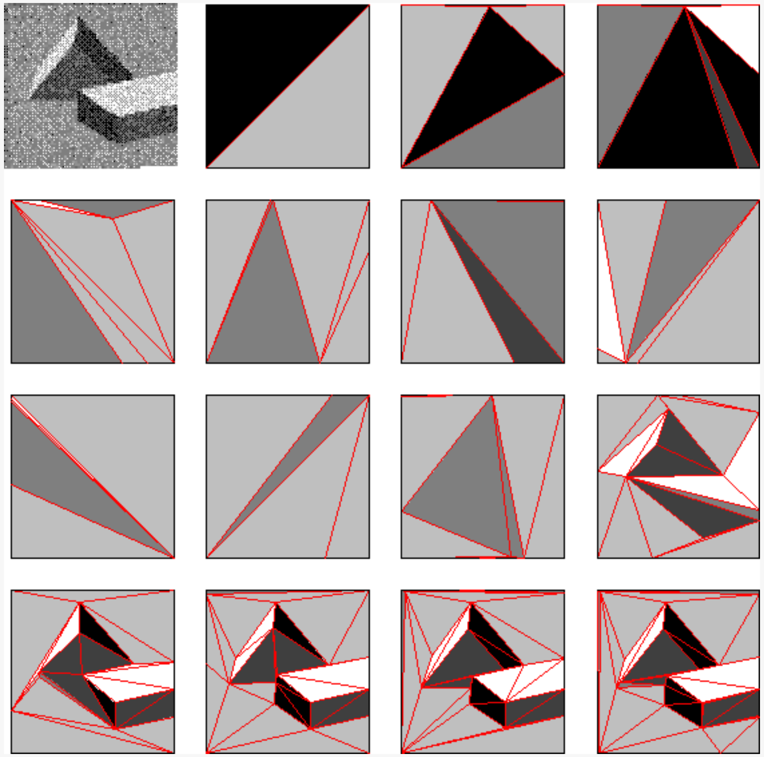






This algebraic representation is very difficult for MCMC, but produces great results.

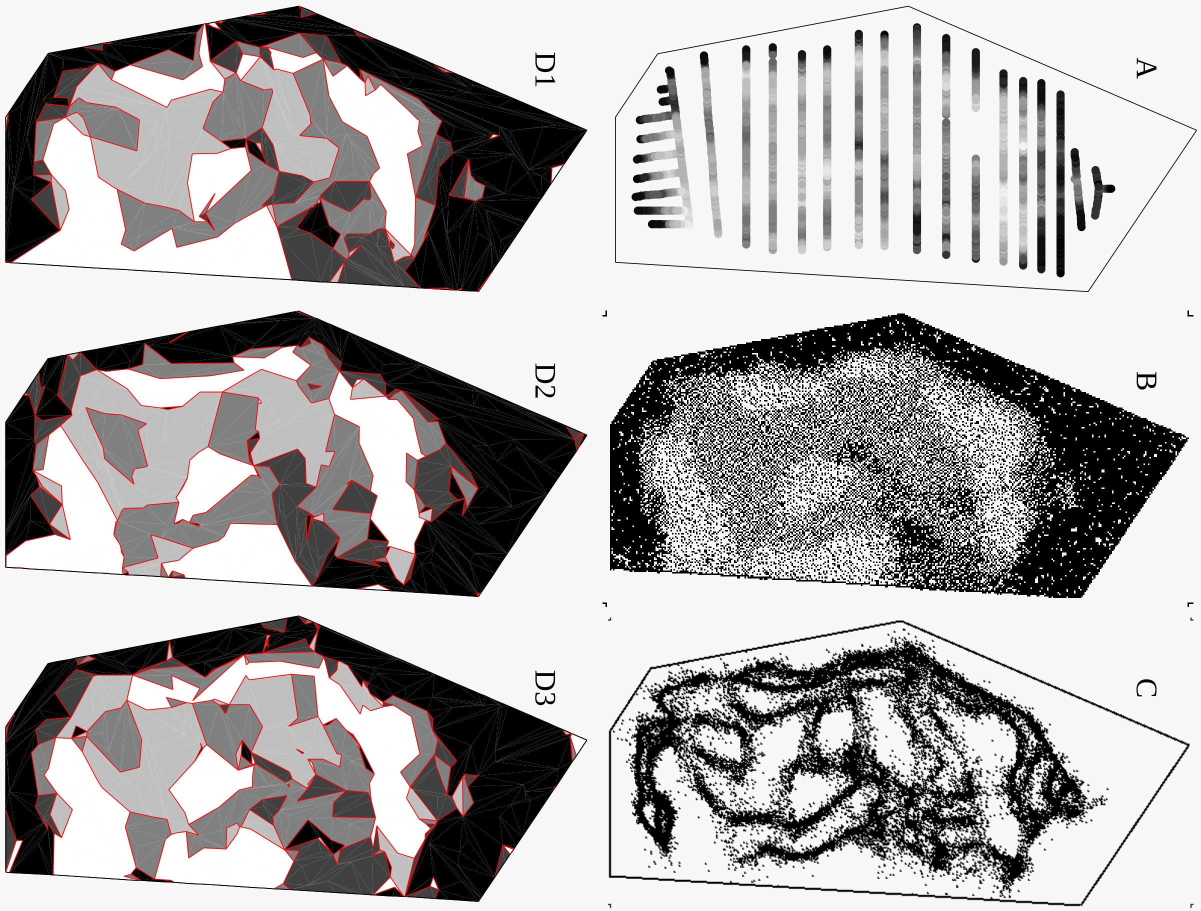
Coloured Continuum Triangulation



$$X = \bigcup_{i=0}^{\infty} \{[0,1] \times [0,1]\}^i \text{, coloured}$$

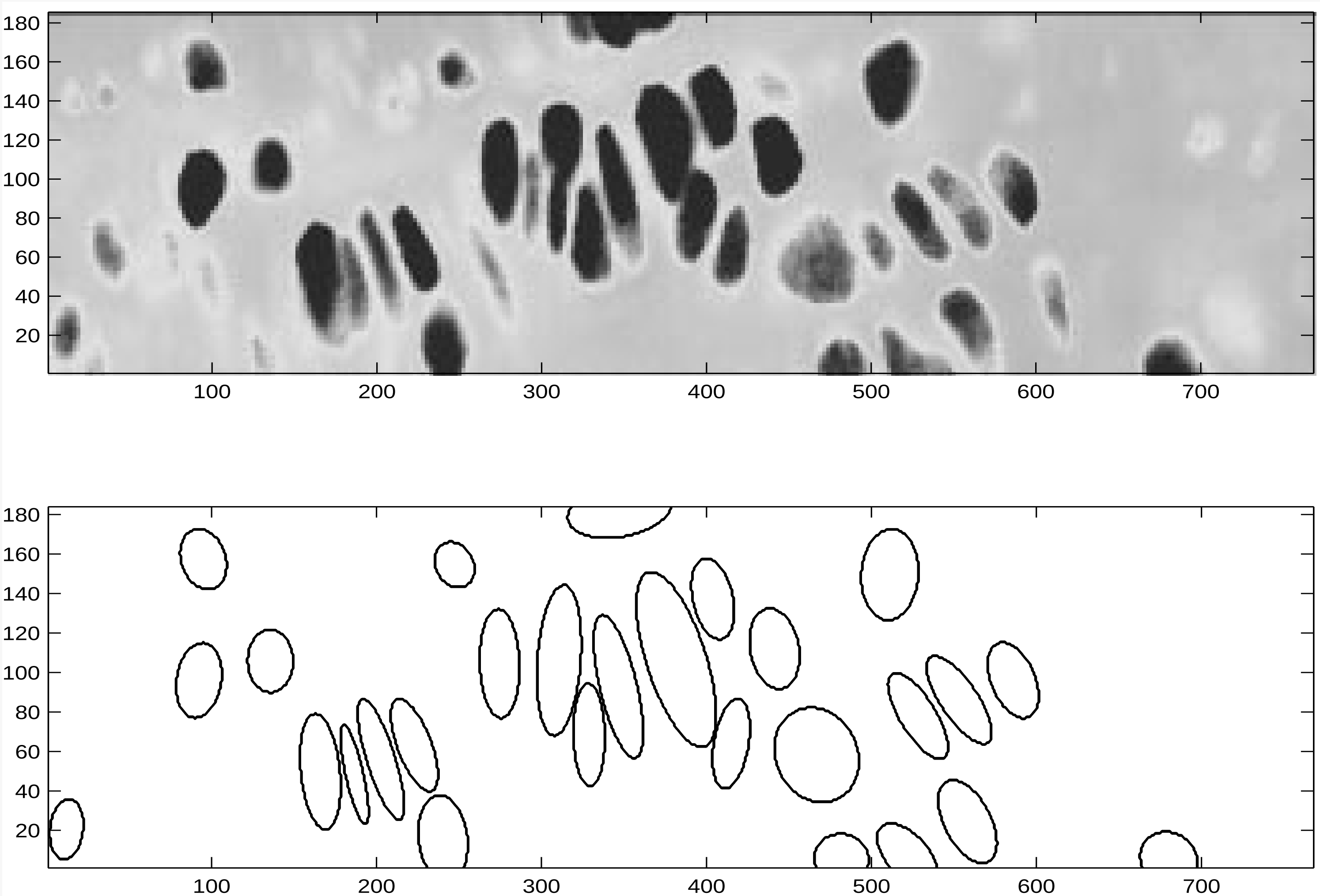
Nicholls 1998 *Bayesian image analysis with Markov chain Monte Carlo and colored continuum triangulation models* JRSSB

Neolithic hill fort (Maori pa)



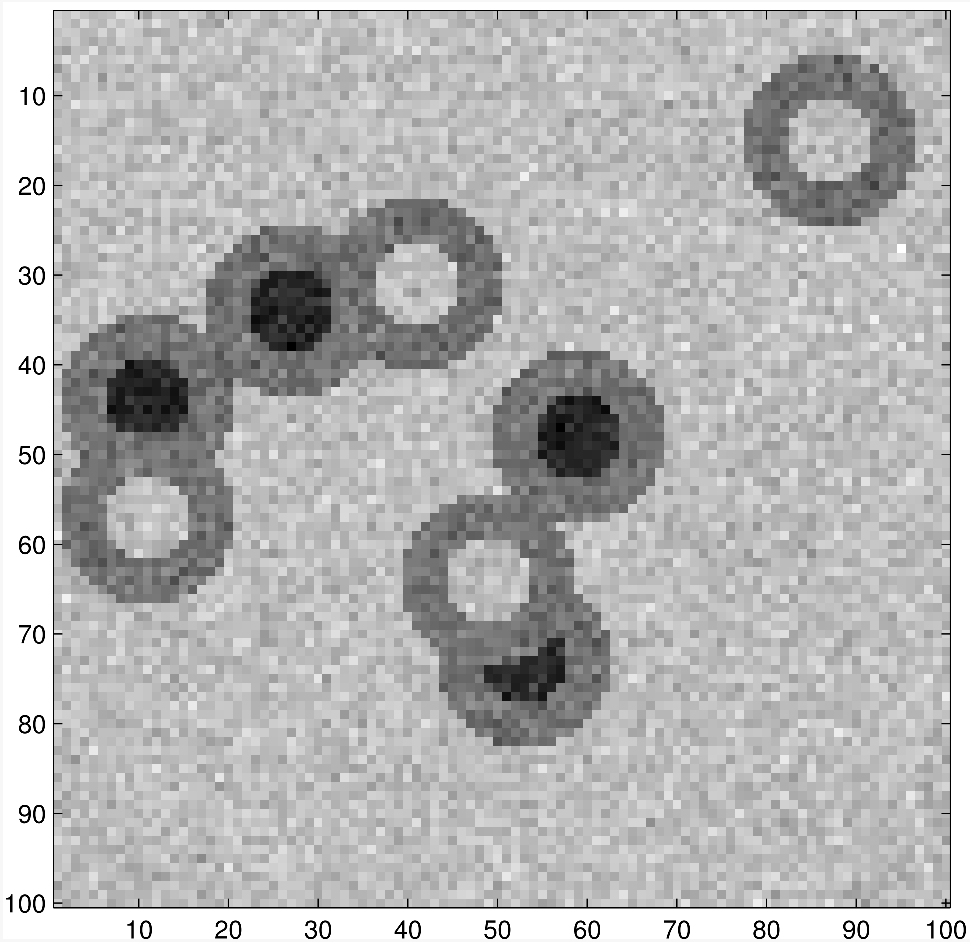
A) data, 1746 resistivity readings, (B) posterior mean resistivity, (C) posterior edge length density, (D1-3) samples from posterior

Marked Point Process

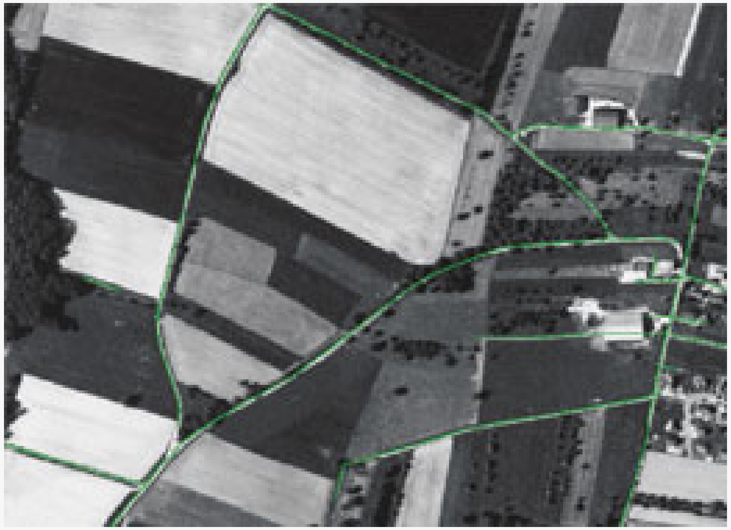


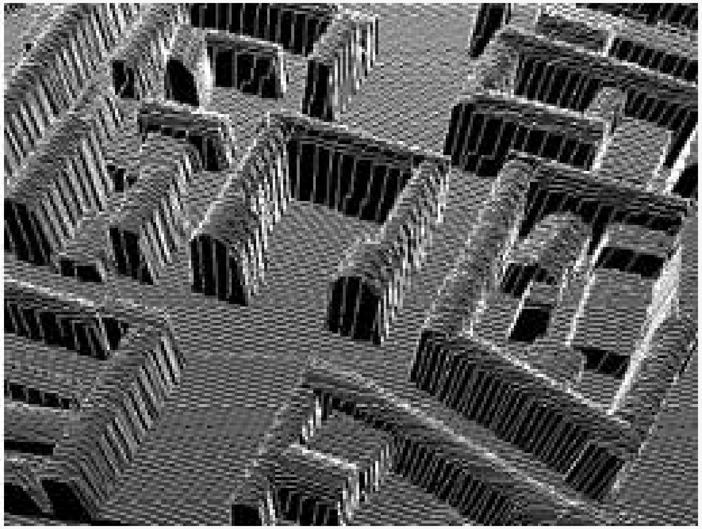
Fahimah Al-Awadhi, Christopher Jennison, Merrilee Hurn 2004 JRSSC (Appl. Statist.)

Computing lab today : count good/bad cells



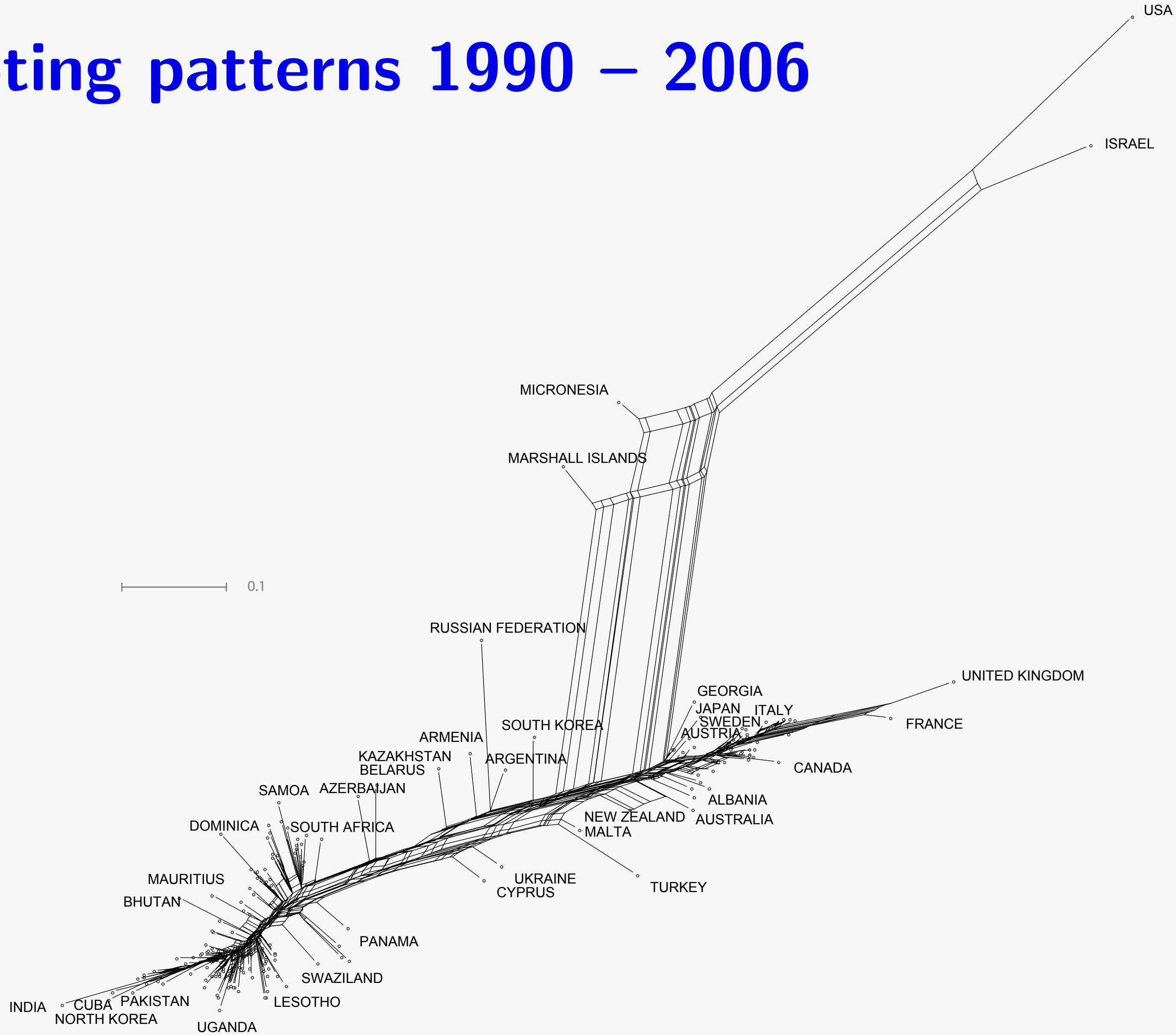
More Exciting State Spaces





Josiane Zerubia, Xavier Descombes, C. Lacoste, M. Ortner, R. Stoica (2000, 2003)

U.N. voting patterns 1990 – 2006



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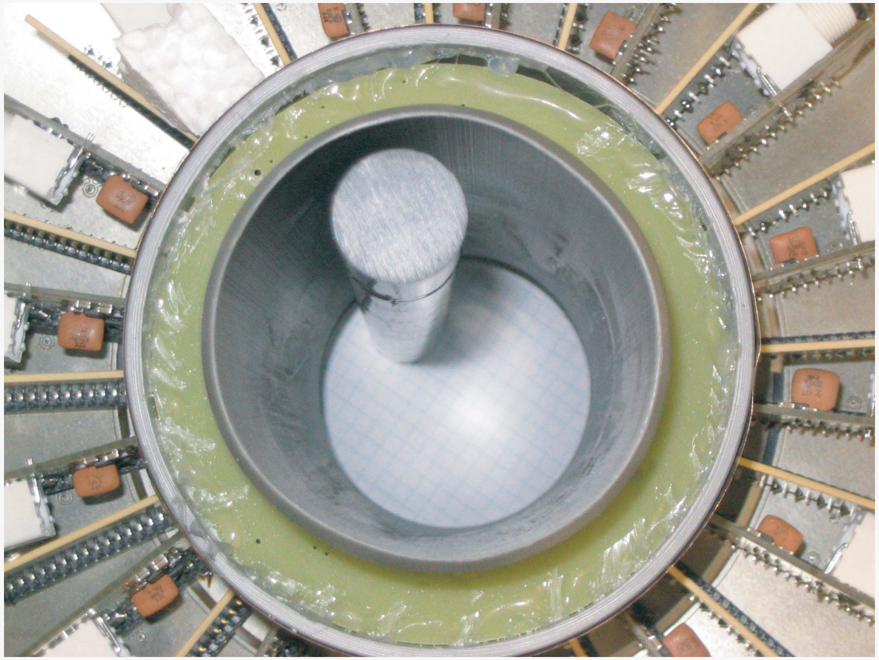
thanks to David Bryant

Electrical capacitance tomography

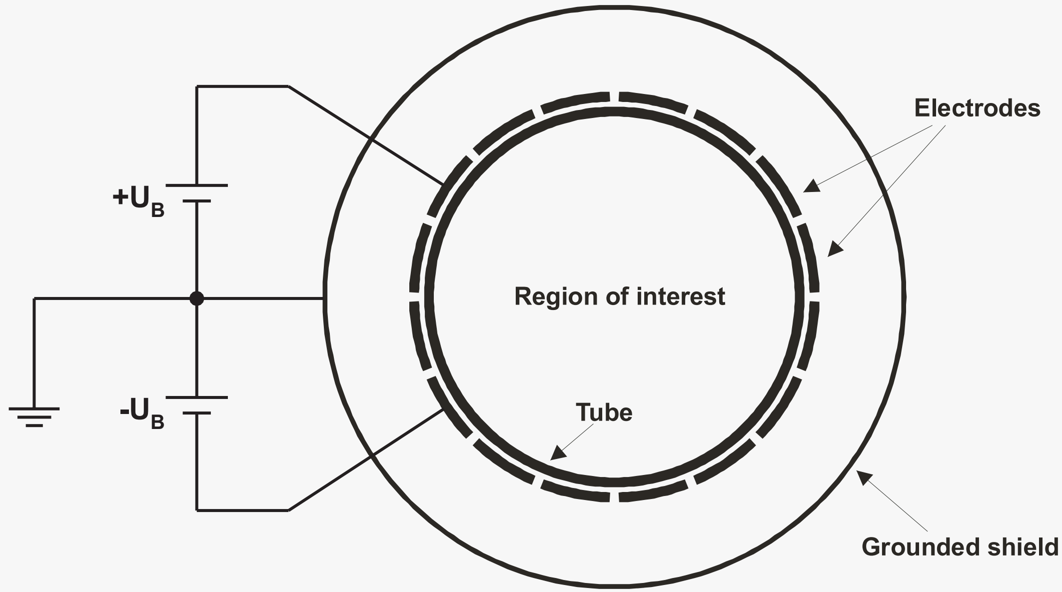


- Measure inter-electrode capacitances (1 fF to 5 pF)

$$\mathbf{q} = C\mathbf{v}$$

- Non-invasively image permittivity ε
- Primarily interested in (2-dim) area of inclusion

ECT measurement system



Assert N_M potential vectors $\mathbf{v}^m = \left\{ v_1^m, v_2^m, \ldots, v_{N_\mathsf{E}}^m \right\}^\mathsf{T}$, for $m = 1, 2, \ldots, N_\mathsf{M}$

Resulting potential fields denoted u^m

Measure vector of (displacement) charges is $\mathbf{q}^m = \left\{ q_1^m, q_2^m, \ldots, q_{N_\mathsf{E}}^m \right\}^\mathsf{T}$

$\mathbf{q}^m$ is a *linear* function of $\mathbf{v}^m$, hence

$$\mathbf{q} = C\mathbf{v}$$

where C is the $N_\mathsf{E} \times N_\mathsf{E}$ matrix of trans-capacitances.

Forward map G

ECT

$$\nabla \cdot (\varepsilon \nabla u) = 0 \qquad \text{in } \Omega \cup \Omega_{\mathsf{E}}$$

$$u|_{\partial\Omega_k} = v_k \qquad k = 1, 2, \ldots, N_{\mathsf{E}}, S$$

Measured charge related to fields by

$$q_k = \int_{\partial\Omega_k} \varepsilon \nabla u \cdot \mathbf{n}\, dl, \qquad k = 1, 2, \ldots, N_{\mathsf{E}}$$

EIT

$$\nabla \cdot \sigma \nabla u = 0 \qquad \text{in } \Omega$$

$$\int_{e_l} \sigma \frac{\partial u}{\partial n} dS = I_l$$

$$\sigma \frac{\partial u}{\partial n}\bigg|_{\partial\Omega \setminus \bigcup_l e_l} = 0$$

$$\left(u + z_l \sigma \frac{\partial u}{\partial n} \right)\bigg|_{e_l} = U_l$$

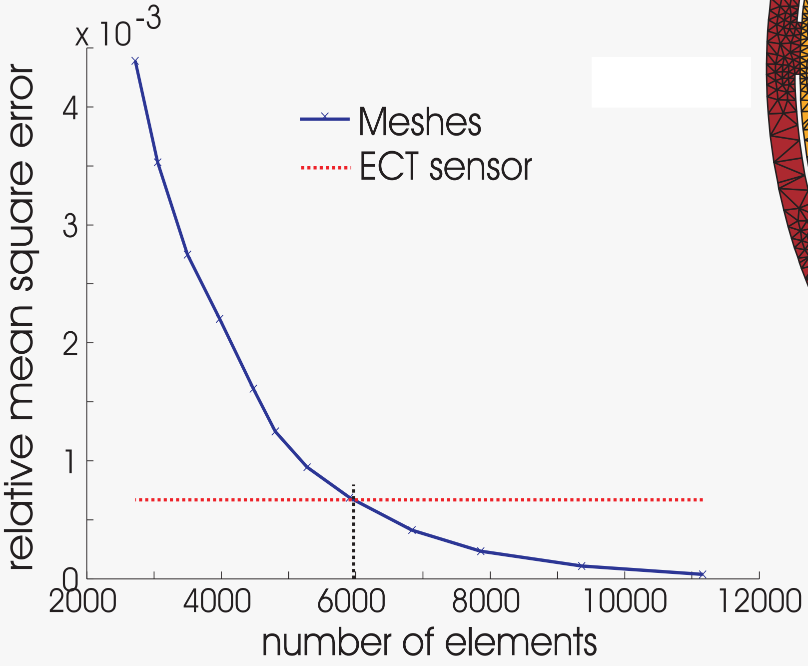
Data simulation requires $N_{\mathsf{M}} \sim 16$ solves of the Dirichlet (Neumann) BVP.

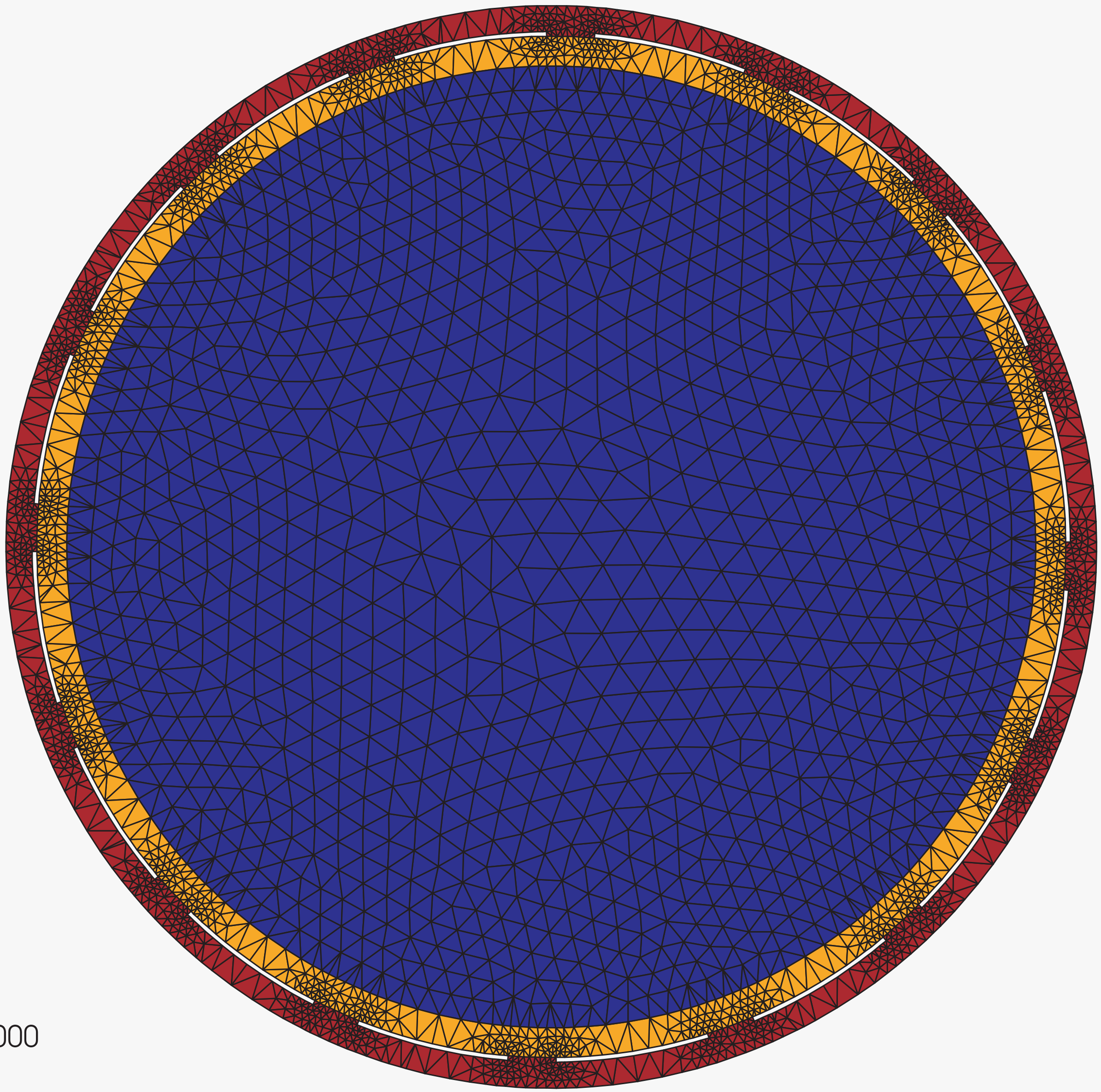
SNR of 1:1000 provides 105 measurements + 5 per factor of 10 (further measurements give $\sqrt{n}$ noise improvement). Correlation = 1-0.

Big names (Ohm, Kirchhoff, Laplace, Maxwell), but the biggest source of error!

$$\pi(\varepsilon \mid \mathbf{q}) \approx \pi_{\mathrm{n}}(\mathbf{q} - G(\varepsilon))\pi_{\mathrm{pr}}(\varepsilon)$$

FEM Mesh for ECT



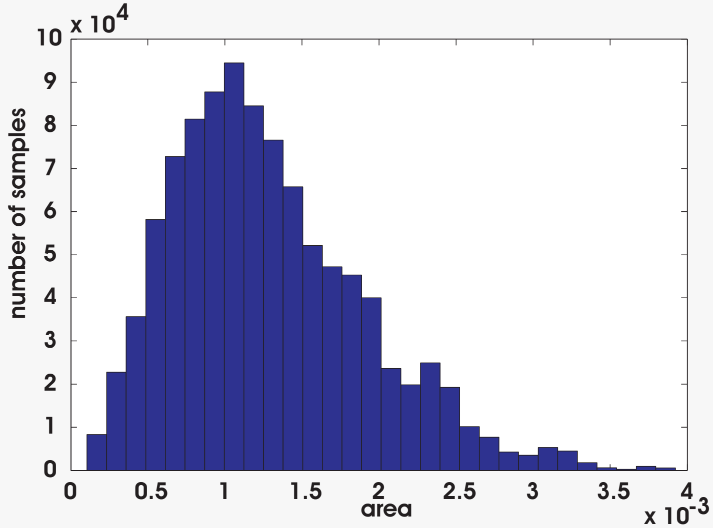


Represent boundary by implicit RBF (or polygon)

Represent boundary by N point implicit RBF x

Naive prior uniform in node position: $\pi_{\mathrm{pr}}(x) = I(\text{allowable contour})$

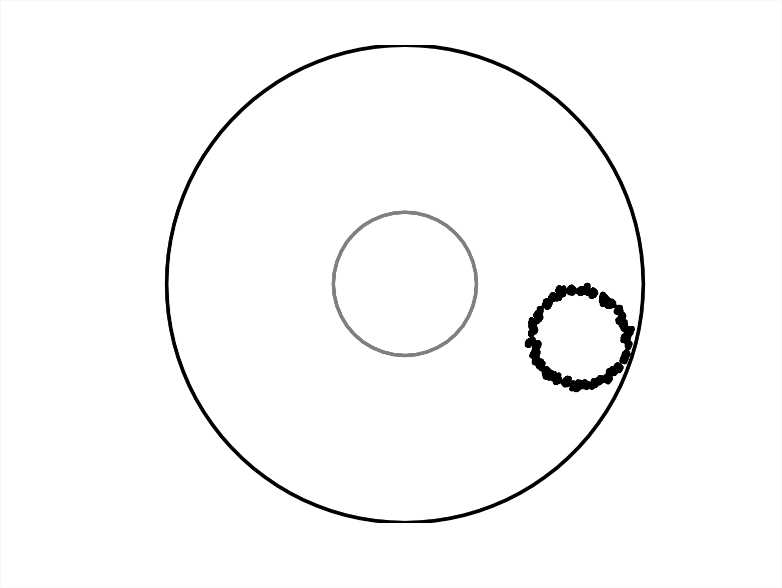
For large area $\pi_{\mathrm{pr}}(\text{area}) \propto (\text{area})^{-1/2}$

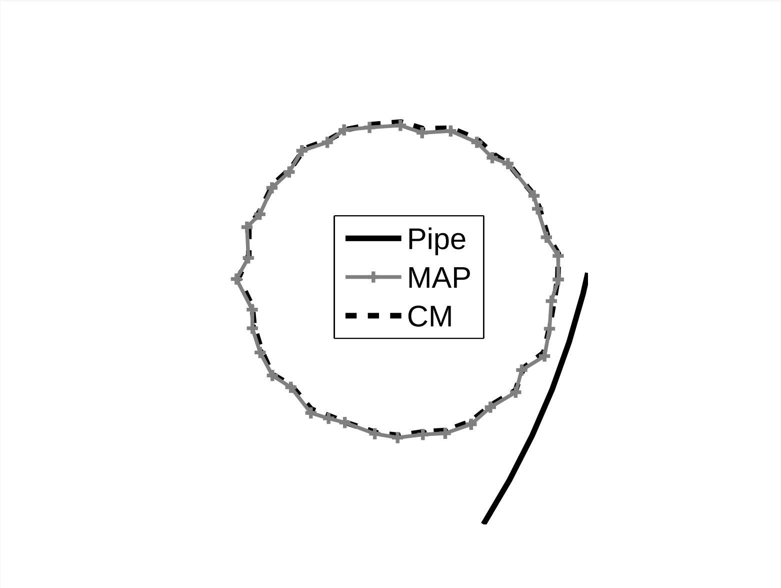


Specify a prior explicitly in terms of area $\Gamma(x)$ and circumference $c(x)$

$$\pi(x) \propto \exp\left\{-\frac{1}{2\sigma_{\mathrm{pr}}^2}\left(\frac{c(x)}{2\sqrt{\Gamma(x)\pi}} - 1\right)\right\} I\left(x\right)$$

Posterior estimates (measured data)





Quantities	true values	mean	standard deviation	IACT
x-coordinate of center [m]	–	3.71×10^{-2}	2.32×10^{-5}	5.89×10^{2}
y-coordinate of center [m]	–	-1.14×10^{-2}	3.02×10^{-5}	4.65×10^{2}
Area Γ [m^2]	3.14×10^{-4}	3.13×10^{-4}	6.88×10^{-6}	1.10×10^{3}
Circumference c [m]	6.28×10^{-2}	6.24×10^{-2}	1.57×10^{-4}	1.88×10^{3}
Log-likelihood	–	-46.10	1.72×10^{-1}	3.99×10^{2}