1. (a) For a system with Hamiltonian $H = H_0 + H_I$ where H_I represents the interaction, give the equations of motion for states and operators in the

   (i) Schrödinger picture;

   (ii) Heisenberg picture;

   (iii) Interaction picture.

   (b) An atom in an optical cavity is interacting on resonance with a single cavity mode of the Electromagnetic field. The atom-light system is described by the Jaynes-Cummings Hamiltonian

$$H = \hbar\Omega\left(a^\dagger a + \tfrac{1}{2}\sigma_z\right) + \hbar\kappa\left(a\sigma_+ + a^\dagger\sigma_-\right) \equiv H_0 + H_{Int},$$

   where the photon operators have Bose commutation relations $[a, a^\dagger] = 1$, and the two-level atom is described by the Pauli matrices, and $\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y)$ are raising and lowering operators.

   (i) Show that $[H_0, H_{Int}] = 0$, and thus show that the interaction Hamiltonian in the interaction picture is unchanged: $H_{Int,I} = \hbar\kappa\left(a\sigma_+ + a^\dagger\sigma_-\right)$.

   (ii) Working in the interaction picture, consider an initial state $|\psi, 0\rangle = |e, n\rangle \equiv |e\rangle \otimes |n\rangle$, where the atom is in the excited state $|e\rangle$ and the cavity mode is in the number state $|n\rangle$. Expand the quantum state as

$$|\psi, t\rangle = \lambda(t)|e, n\rangle + \mu(t)|g, n+1\rangle,$$

   and find and solve the equations of motion for $\lambda(t), \mu(t)$, and thus find $|\psi, t\rangle$.

   (iii) Find the probability that the system is in the state $|e, n\rangle$ at time t.

   (c) Consider now the situation where the EM mode is initially in a coherent state

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

   so that $|\psi, 0\rangle = |e, \alpha\rangle$.

   (i) Write down an expansion of the state ket in terms of the number state solutions found above.

   (ii) Give an expression, in terms of the density matrix of the total system $\rho(t) = |\psi, t\rangle\langle\psi, t|$, for the probability $p_e(t)$ that the atom is in the excited state at time t, irrespective of the EM mode state. Evaluate this expression to find $p_e(t)$, and write your answer in terms of the mean photon number for $|\alpha\rangle$.

   (iii) Express $p_e(t)$ in terms of a time independent term and a sum of oscillating terms. Interpret your result.