2. (a) For a system with Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_{\text{Int}}$ where $\hat{H}_{\text{Int}}$ represents the interaction, briefly comment on the evolution of quantum states and operators in the

 (i) Schrödinger picture;

 (ii) Heisenberg picture;

 (iii) Interaction picture with respect to $\hat{H}_0$.

(b) Given a time-dependent Hamiltonian $\hat{H}(t)$, and time-independent operator $\hat{X}_S$ in the Schrödinger picture, show that the formal solution

$$\hat{X}_H(t) = \mathbf{T}\left[\exp\left(\frac{i}{\hbar}\int_0^t \hat{H}(t')dt'\right)\right]\hat{X}_S\,\mathbf{T}\left[\exp\left(-\frac{i}{\hbar}\int_0^t \hat{H}(t')dt'\right)\right]$$

satisfies the Heisenberg equation of motion for $\hat{X}_H(t)$, where $\mathbf{T}$ time-orders operators into chronological order.

(c) A single mode of the electromagnetic field evolves according to the Schrödinger picture Hamiltonian

$$H = \hbar\omega\hat{a}^\dagger\hat{a} + i\hbar g(t)(\hat{a}^\dagger - \hat{a}) \tag{1}$$

where $\hat{a}^\dagger$, $\hat{a}$ are creation and annihilation operators for the mode with $[\hat{a}, \hat{a}^\dagger] = 1$, and $g(t)$ is a real-valued function of time.

 (i) Give the Heisenberg equations of motion for the creation and destruction operators. Solve the equation of motion for the destruction operator $\hat{a}(t)$, and show explicitly that the commutation relations are preserved, i.e., the solutions satistfy

$$[\hat{a}(t), \hat{a}^\dagger(t)] = 1.$$

 (ii) If the state of the system is the vacuum state $|0\rangle$ of the *initial* creation and destruction operators (i.e., if $\hat{a}(0)|0\rangle = 0$) show that this state satisfies

$$\hat{a}(t)|0\rangle = \beta(t)|0\rangle$$

where

$$\beta(t) = \int_0^t dt' g(t')e^{-i\omega(t-t')}.$$

 (iii) Using the relationship between the Schrödinger and Heisenberg pictures, show that in the Schrödinger picture, the quantum state at time t is the coherent state $|\beta(t)\rangle$; where the coherent state $|\alpha\rangle$ is defined as the eigenstate of the annihiliation operator:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

 (iv) Give an example of a physical system that may be described by the Hamiltonian (1).